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Abstract

This work provides an extensive overview of the input-output (IO) literature, both theoretical and empirical, dealing with the inherent IO data uncertainty issues. The survey is carried out on the basis of a specific uncertainty technique used, rather than taking a chronological overview approach, which also allows for easier comparisons and linking of the outcomes of the individual contributions. Thus, we discuss the literature within seven methodological blocks (sections), which include deterministic error analysis, econometric and other (non-Bayesian) statistical approaches, random error analysis and probabilistic approach, full probability density distribution approach, Monte Carlo analysis, Bayesian approach, and other techniques. Within each section, the literature on a certain topic is reviewed in its historical context, which helps to clarify the state of the art. Our main findings from this survey, related discussions, final remarks and observations are given in the concluding section.

Keywords: *input-output uncertainty, deterministic and random error analysis, stochastic input-output analysis, Monte Carlo simulations, Bayesian approach*

JEL Classification: *C67, D57, R15*

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UNCERTAINTY TREATMENT IN INPUT-OUTPUT ANALYSIS

1. INTRODUCTION

1.1 Input-output uncertainty and the scope of this survey

The problem of uncertainty in input-output (IO) data, which may have real consequences in certain cases, is and most probably will always be present. Leontief himself stated that '[f]irst of all, there is the immediate problem of the numerical accuracy of the individual entries' in IO tables, but recognized that '[t]he question of numerical accuracy unfortunately cannot be answered as simply and directly as it can be posed. In order to know how inaccurate are the figures presented in published tables, one would have to possess the true measures of the magnitudes in question; but if these were available, they certainly should have been used in the first place' (Leontief, 1955, pp.9-10). Similarly, Miernyk (1976, p.54) observed that '[t]here is no way of measuring differences between "true" coefficients and those calculated from survey data, since the true coefficients must remain forever unknown. But we can measure the representativeness of sectoral samples … and careful craftsmen will do everything they can to avoid introducing bias when estimating interindustry transactions from sample data'.

In economic modeling, in general, uncertainty (of final modeling outcomes) is an extremely important issue, and has numerous sources. One of such source includes errors in data, which is particularly important source of uncertainty in the field of IO analysis due to its reliance in practice on diverse-sized IO and supply and use tables. These data uncertainties are caused by many factors, such as sampling errors, measurement errors, errors generated during the IO data compilation process, confidentiality issues, aggregation errors, prices and deflation practices, and reporting errors. Within a multi-regional IO setting that covers several national or regional economies, additional sources of errors arise, such as concordance to a common industry and/or product scheme, monetary exchange rates (i.e. market exchange rates vs. purchasing power parities), and treatment of the rest-of-the-world region (see e.g. Weber, 2008). Further, there is the problem of errors associated with trade flows estimation and the need to separate flows of intermediate from final goods.

In this chapter, we provide an extensive survey of the IO literature focusing on the issue of uncertainty in IO analysis.[1] This body of literature has a rather long history, and includes an enormous number of theoretical and/or empirical studies. As mentioned in Miller and Blair (2009), the IO literature 'on the impact (transmission, propagation) of errors or changes or uncertainty in basic input-output data on the model outcomes … has appeared under a variety of titles, [such as] "probabilistic" or "stochastic" input-output, "error" analysis and "sensitivity" analysis, and so on' (p.567). Thus, it is unreasonable and not useful anyway to overview each and every study of this literature, and therefore the chapter will survey in detail the most relevant and important contributions.

In particular, the following three bodies of literature fall outside the scope of this overview. First, we do not review a large body of literature on the accuracy of non-survey methods used to build missing regional, national, and/or inter-regional IO tables or supply and use tables (SUTs). Often in these studies, the outcomes of different non-survey techniques are compared to those of their survey-based counterparts, on the basis of which conclusions regarding superiority of (or unacceptability of using) certain methods are made. For example, Morrison and Smith (1974), McMenamin and Haring (1974), and Round (1978) are examples of earlier studies in this line of research, while some important recent contributions include Jackson and Murray (2004), Temurshoev, Webb and Yamano (2011), Lehtonen and Tykkyläinen (2014), Flegg and Tohmo (2014), and Rueda-Cantuche (2015). Although estimation errors due to the use of specific (partial) non-survey methods contribute to the IO data uncertainty, it is not the aim of this chapter to discuss (dis)advantages of certain (partial) non-survey methods (for this the reader is referred to the above-mentioned papers).[2]

Secondly, the literature on the impact of (sectoral and/or regional) aggregation errors, again though relevant to the uncertainty issue of IO models outcome, will not be surveyed here. It is clear that the more disaggregated the IO tables or SUTs, the greater the opportunity to better account for inherent heterogeneities in a broad sense (e.g. in terms of linkages, by-products, prices) between sectors and/or commodities. Thus, usually preference would be for maximally disaggregated data, provided that this

[1] Manski (2015) raises the issue of uncertainty, in general, in the reports of statistical agencies and discusses 'strategies to mitigate misinterpretation of official statistics by communicating uncertainty to the public'. In particular, he distinguishes between transitory statistical uncertainty, permanent statistical uncertainty, and conceptual uncertainty.

[2] Full non-survey methods have certainly their drawbacks compared to partial non-survey techniques. See e.g. Miernyk (1976) and Round (1983) for early criticism of 'wholly nonsurvey methods' (Round, 1983, p. 209).

does not sacrifice the quality of the data in the first place. Some of the earlier and recent relevant contributions include Doeksen and Little (1968), Morimoto (1970), Hewings (1972, 1974), Kymn (1990), Lahr and Stevens (2002), Andrew, Peters and Lennox (2009), Lenzen (2011), and Bouwmeester and Oosterhaven (2013).

And, thirdly, we do not consider a large and growing literature in Life-Cycle Assessment (LCA) that uses Monte Carlo techniques in dealing with uncertainties, where researchers use generalized IO systems in order to link physical satellite accounts with monetary tables. For a sample of related discussions, see Lenzen (2000), Hondo, Sakai and Tanno (2002), Peters (2007), and Heijungs and Lenzen (2014).

We are aware of only three studies, namely Jackson and West (1989), Kop Jansen (1994) and Gurgul (2007), that review the literature of stochastic IO analysis. These surveys do not provide such an extensive discussion of the literature as we do in this chapter. Rather than reviewing papers in chronological order, our overview approach is based on the discussions of the relevant studies according to the techniques used; this approach facilitates easier comparisons and linking the results of separate contributions. For this reason, we have seven separate sections according to the tools researchers have been using to tackle the IO uncertainty problems. These are deterministic error analysis, econometric and other (non-Bayesian) statistical approaches, random error analysis and probabilistic approach, full probability density distribution approach, Monte Carlo analysis, Bayesian approach, and other techniques. Within each block the review of the literature is carried out in its historical context, which consequently clarifies the state of the art. However, we believe that the notion of the ‘state of the art’ is *relative*, in particular in economics, in the sense that sometimes (if not often) old ideas, research focus, and/or approaches suddenly become core or mainstream for the latest research. It is from this perspective that we also discuss the sequential developments of the topics within each covered sections, taking sort of ‘history of thought approach’.

1.2 Roadmap and the macro-overview of the literature

Given that the scope of this review is rather broad, in what follows we provide a roadmap for a reader to facilitate processing of the presented material. Some might use this roadmap to jump immediately to the section(s) they are mostly interested in.

For this purpose, also the common mathematical notations used throughout this chapter are introduced in the following subsection.

Section 2 reviews IO deterministic error analysis, starting with a brief discussion of the literature providing the appropriate mathematical apparatus. The error analysis literature's focus is mainly on the open Leontief system, where input coefficients and final demand are exogenous variables, and gross outputs, employment, income and, in general, the so-called 'importance function' are endogenous variables. The main issue essentially is how errors in exogenous data are transmitted into the endogenous variables and into various (e.g. output, employment) multiplier matrices and/or vectors. This is not a straightforward task, given that e.g. the relation between intersectoral input coefficients and multiplier matrix elements is non-linear. There are also few studies that consider partially closed Leontief system, where households are 'endogenised' along with industries. In these studies, the consequences of introducing errors into the vectors of consumption coefficients and labor input coefficients are additionally considered.

In Section 3, econometric and other non-Bayesian statistical approaches to estimation of direct input coefficients, direct output coefficients and IO multipliers are surveyed. It reveals that in the time-series approach aggregate (national accounts) data were used, while cross-sectional approach is based on establishment data. The last approach better fits for the purpose of estimation of input and/or output coefficients and IO multipliers on, at least, two grounds: firstly, structural change and product mix variations over time may make the estimates from time series approach meaningless, and secondly, large sample size of establishment data avoids the problem of insufficient degrees of freedom in regressions implementation. Given that establishment data may include data on both firms' purchases and sales, the two data need to be reconciled as usually the two types of data are inconsistent. We do not discuss reconciliation problem in detail, but as briefly discussed in Section 3, reconciled estimators of input and output coefficients deal with this issue explicitly. Statistical approaches to supply and use tables (SUTs) estimation or multipliers estimation based on SUTs are reviewed as well.

Section 4 focuses on random error IO analysis, where input coefficients, final demand, intermediate transactions, and/or primary inputs coefficients are considered as random variables. The main focus of this literature is on finding, in the majority of cases analytically, how such randomness affects the expected values and/or variance of

Leontief inverse, gross outputs, and IO multipliers, which allows reporting the outcomes of interest with the relevant confidence intervals. One of the extensively covered core issues is the problem of over- or under-estimation of the Leontief inverse elements in practice compared to their ‘true’, free-of-errors counterparts. Thus, also the relevant biases in IO multipliers and IO solution are considered.

Section 5 covers the so-called ‘full probability density function (pdf)’ approach, proposed by Jackson (1986). It is in some sense closely related to the random error analysis, specifically when explicit densities for the considered errors are assumed. However, the foundation of the full pdf approach is quite distinct from that of the random error analysis: even in the absence of errors, there are ranges of probable outcomes of input coefficients, multipliers, and outputs due to the expected systematic variation in establishment level production coefficients because of the variety of industrial, institutional, and location factors. For example, such variables as technology, labor productivity, prices, age of capital stocks, product mix, firm’s size, market structure, all of which have different spatial characteristics, contribute to the probabilistic nature of interindustry interactions.

Monte Carlo simulation techniques have been extensively used in dealing with the IO uncertainty issues, which is the subject of study of Section 6. The usefulness of Monte Carlo analysis is especially important when no general analytical results on the features of distributions of variables of interest exist. The relevant surveyed literature focus is rather diverse, and include e.g. distribution characteristics of gross outputs, relative importance of regional purchase coefficients vs. technical coefficients, accuracy importance of large vs. small IO coefficients, output multiplier matrix bias, carbon footprints uncertainties, significance of temporal changes of the total forward and backward linkages, relative importance of different types of uncertainties in integrated econometric IO models and climate change models, and sensitivity evaluations in equilibrium analysis.

Section 7 reviews Bayesian approaches to uncertainty treatment in IO analysis. The topics include compilation of national accounts – including SUTs, IO data updating and balancing problems, Bayesian integrated IO and econometric modelling, and estimation of IO multipliers and of intercountry feedback-spillover effects. This is a recent line of research in IO analysis, hence the number of papers surveyed is also limited.

IO-related studies using methodologies other than those covered in Sections 2 to 7 are briefly discussed in Section 8. This section is mainly added for completeness purposes, as presentation of the details of the approaches are (largely) missing, but nevertheless it provides an overview of other techniques that could also be potentially useful in IO uncertainty treatment. Finally, Section 9 discusses some of the main findings of this literature survey, and related remarks and observations on the importance of uncertainty treatment in IO analysis and potentially fruitful future research directions.

Table 1 provides a concise summary of the papers surveyed according to the above-mentioned methodological blocks/sections.[3] Given that for each study Table 1 also provides information about the uncertainty source and/or study focus together with a brief summary of the paper's results or extra relevant information, it is expected that this overview table will be useful for the reader as an extra tool/roadmap that facilitates going smoothly through the entire material. In addition, it provides a concise macro-overview of the literature and the corresponding most important results.

Table 1: Summary of the reviewed literature.

Uncertainty technique	**Uncertainty focus/source**	**Main contributions with brief summary or extra clarifying information**
1) Deterministic error analysis	Direct input coefficients (and final demand and/or value added if the study year is marked with an asterisk)	Dependent errors with the following correlation structures: -- Errors in one entry, one row and two rows: Evans (1954*), West (1982). -- General error structure with a common maximum bound: Christ (1955). -- General error structure: Henderson (1955), Park (1973*), Sebald (1974), Bullard and Sebald (1977), West (1982), Sonis and Hewings (1989, 1992, 1995; fields of influence approach), Dietzenbacher (1990; an eigenvector approach).
2) Econometric and other (non-Bayesian) statistical approaches	Direct input and/or output coefficients	-- Stone (1955), Briggs (1957), Klein (1974): use of time series data; OLS, maximum likelihood method. -- Gerking (1976a,b; 1979a), Hanseman & Gustafson (1981): use of cross-sectional data; OLS, 2SLS, Wald-Bartlett estimator, Durbin's method, reconciled estimation. -- Kockläuner (1989): cross-sectional approach, randomly varying input and output coefficients, perfect aggregation structure, reconciled estimators in finite populations.
	IO multipliers	-- Braschler (1972): OLS of total employment on sectoral exports to estimate employment multipliers, cross-section of county level observations (similar to economic base studies). -- Ten Raa and Rueda-Cantuche (2007): estimating IO multipliers using SUTs data at establishment level. Related papers dealing with economy-wide SUTs data include Rueda-Cantuche and Amores (2010), Rueda-Cantuche (2011; termed the approach as SUBE approach), and Rodrigues and Rueda-Cantuche (2013).
	Supply and use tables/flows (SUTs)	-- ten Raa and van der Ploeg (1989): using SUTs data variances and a maximum likelihood approach, obtain new, updated SUTs that are consistent with non-negative IO coefficients under commodity technology model, sensitivity of input coefficients with respect to SUTs entries is examined. See also ten Raa (1988), Mattey and ten Raa (1997), and Rueda-Cantuche and ten

[3] However, there are still other papers discussed in the text that are not included in Table 1. These are studies that fall outside the scope of this survey (cited in Section 1), and studies that do not deal directly with the IO uncertainty issues, e.g. mathematical contributions used for deterministic error analysis are not mentioned.

		Raa (2013).
3) Random error analysis and probabilistic approach	Direct input coefficients (and final demand if the study year is marked with an asterisk)	-- Evans (1954, section 4): a row of input matrix subject to independent random errors, coefficient of variations of gross outputs are (much) less than those of input coefficients. -- Quandt (1958): independent and symmetric errors (2 sector economy), expressions for (co)variances of outputs, establishes confidence intervals for gross outputs with critical values of normal distribution. -- Simonovits (1975): overestimation of Leontief inverse with independent input coefficients, both under- and over-estimation of Leontief inverse elements with dependent input coefficients (error rectangle). -- Goicoechea and Hansen (1978*): reformulate Leontief system in probability terms that allow for any type of pdf's choice for input coefficients and final demand, nonlinear system of equations including the uncertainty level, appropriate for sensitivity/error analysis. -- Lahiri (1983*): overestimation of Leontief inverse with biproportionally stochastic IO coefficients; underestimation of gross outputs vector in non-linear IO system with stochastic final demand. -- Flåm and Thorlund-Petersen (1985*): overestimation of Leontief inverse and gross outputs when input coefficients and final demands are non-negative moment-associated random variables. -- Lahiri and Satchell (1985): over- or under-estimation of Leontief inverse with errors in prices and uniform errors, both causing randomness in input coefficients. -- West (1986): normally and independently distributed errors, closed-form expressions for IO multipliers density, multipliers are underestimated, multiplier distribution is positively skewed. -- Fox and Quirk (1985): derive the general formula of the probability density function (pdf) of the Leontief inverse for given pdf of the input coefficients. (For further relevant details, see Temurshoev, 2015b.) -- Lahiri and Satchell (1986): validate Simonovits (1975) and Lahiri (1983) results under more general assumption on errors. -- ten Raa and Steel (1994): criticism of West (1986), IO coefficients are assumed to have independent beta distributions defined on the unit interval. -- Kop Jansen (1994): approximating formulas for the first- and second-order moments of the Leontief inverse under the assumption of independently distributed IO coefficients. -- ten Raa and Kop Jansen (1998): first-order approximation formulas of the bias and sensitivity (or covariance) of the Leontief inverse entries (hence IO multipliers) in a generalized setting of dependent (and independent) stochastic IO coefficients.
	IO transactions /flows (and primary input coefficients and/or final demand, if the study year is marked with an asterisk)	-- McCamley, Schreiner and Muncrief (1973): approximating formula for reliabilities (variances) of IO [employment] multipliers with dependent random IO transactions. -- Lahiri and Satchell (1986): single error rectangular assumed on intermediate flows, see above. -- Dietzenbacher (1988*): arbitrary series of error rectangles, each nonzero row and column of the bias matrix of the Leontief inverse must contain elements with opposite signs; considers also error couple structure when only the condition of consistency of total inputs and total outputs is imposed. -- Dietzenbacher (1995*): the assumption of independently distributed and unbiased inputs coefficients imply that underlying transactions table entries exhibit only positive biases in their row sums; thus stochastics should be imposed on transactions table instead; in the last case, the biases within each row (and column) of the multiplier matrix cancel each other out in the sense that their weighted average is zero; the conjecture of unbiased multiplier estimates is not true in general.
4) Full distribution (full *pdf*) approach	Direct input coefficients	-- Jackson (1986): input coefficients with full pdf's representing all possible values of the coefficients, not only observed ones, and their probabilities that take account of industrial, institutional and location factors. See also Jackson and West (1989), and Jackson (1989).

5) Monte Carlo analysis	Direct input coefficients	-- Quandt (1959): lognormal distribution adequately describes the solution of the Leontief system (3 sector economy). -- Stevens and Trainer (1980), Park et al. (1981): errors in RPCs are more important for the accuracy of outputs and multipliers in regional IO studies than errors in technical coefficients, used multiplicative error structure. -- Jensen and West (1980): experiments show that larger coefficients have dominant role in multipliers formation; thus, for their overall accuracy, the large IO coefficients need to be estimated as accurately as possible. -- Garhart (1985): both RPCs and technical coefficients, particularly significant ones, are contributing to the inaccuracies of outcomes in regional IO analysis, used hybrid (multiplicative & additive) error structure. -- ten Raa and Steel (1994): see above, Leontief inverse and multipliers moments are derived using Monte Carlo simulations. -- Kop Jansen (1994): using Monte Carlo simulations, recommends applying either West's (1986) or Kop Jansen's (1992, 1994) approximating formulas of IO multipliers moments. -- Bullard and Sebald (1988): theoretical maximum error tolerances in Bullard and Sebald (1977) do not reflect the observed precision in IO calculations, input data uncertainties combine or cancel one another in a manner that holds error magnification to acceptable levels.
	IO transactions matrix or SAM	-- Roland-Holst (1989): disaggregated multiplier estimates are unbiased; if Monte Carlo sample is large enough, these estimates can be significantly more stable than their input matrix counterparts. -- Dietzenbacher (2006): unbiased and independently normally distributed intermediate flows, multiplier estimates are positively biased but the biases are negligibly small, in practice one may proceed as if the multipliers are unbiased. -- Rueda-Cantuche, Dietzenbacher, Fernández and Amores (2013) : imposing stochastics on SUTs, independently normally distributed errors of SUTs elements, randomized SUTs are then RASed to become consistent with each other; biases (both positive and negative) in the Leontief inverse elements exist, but they are negligibly small.
	MRIO transactions, gross outputs, CO2 intensities, final demand (may have different uncertainty sources as well)	-- Lenzen, Wood and Wiedmann (2010): perturbing all components of the IO model according to their relative standard deviations (RSDs), CO_2 multipliers RSDs are combined with RSDs of final demand using error propagation to quantify the uncertainties (SDs) for the UK carbon footprint components. -- Karstensen, Peters and Andrew (2014): how uncertainties in economic data, emission statistics and metric parameters propagate from production- and consumption-based emissions to the global temperature change estimates; at the global and national levels, economic data have a relatively small impact on uncertainty. -- Other studies: Lenzen (2001), Wilting (2012), Miller and Temurshoev (2015).
	Integrated econometric and IO models	-- Rey, West and Janikas (2004): IO coefficient uncertainty, econometric model parameter uncertainty, and econometric disturbance term uncertainty, no decisive conclusion on which source of uncertainty is the most important one.
	Complementarity in IO analysis	-- ten Raa and Shestalova (2015a, 2015b): introduce complemenatarity in IO analysis, which combines the quantity and price IO systems in equilibrium analysis; allows assessing results uncertainty/sensitivity with respect to uncertainties in the structure of the economy.
6) Bayesian approach	SUTs, other basic data, indicator ratios, balances	-- Magnus, van Tongeren and de Vos (2000), Magnus and van Tongeren (2002), van Tongeren and Magnus (2012): Bayesian SNA framework, simultaneous use of basic data, accounting identities, indicator ratios, and reliabilities within one setting, estimating national account variables (e.g. SUTs) and their reliability intervals. -- Lugovoy, Polbin and Potashnikov (2014): update IO tables for selected EU countries and SUTs for Russia using Bayesian methods.
	IO data balancing	-- Rodrigues (2014): application of cross-entropy minimization subject to the first and second moment constraints of the balancing problem; generalized least squares, weighted least squares and biproportional methods are particular cases of the proposed Bayesian approach.
	Integrated models	-- Rickman (2001): Bayesian integrated IO-econometric model of the state of Oklahoma.
	IO multipliers	-- Temurshoev (2012, 2015a): Bayesian approach to estimation of IO multipliers using international SUTs; regression-form equations for IO multipliers estimation for other three transformation models; estimates the global and intercountry feedback-spillover effects.
7) Other techniques	Direct input coefficients	-- Nijkamp, Oosterhaven, Ouwersloot and Rietveld (1992): ordinal input-output analysis, combination of ordinal data techniques and error measurement in IO analysis.
	Input and capital matrices	-- Ćmiel and Gurgul (1996, 1997): forward-lag-type, backward-lag-type and mix-lag-type dynamic IO model, random balanced growth factor.

	Input matrix, other IO data	-- Jerrell (1995), Barker and Rocco (2011): interval arithmetic IO applications. -- Beynon, Munday and Roberts (2005), Beynon and Munday (2007, 2008), Díaz and Morillas, (2010): IO applications of fuzzy systems theory. -- Wu and Chang (2003), Li and Liu (2008): IO applications of grey systems theory. -- Wolff (2005): global robustness measure of IO projections with respect to errors in input matrix; use of the theory of robust systems in linear algebra.

1.3 Notations

We adopt the usual IO convention for matrices and vectors, i.e. matrices are indicated by bold capitals, vectors by bold lowercases, and scalars by italic lowercases. Vectors are columns by definition, and matrix transposition is indicated by a prime. The following common to all sections symbols/notations are used, which are also defined upon their first appearance in the text.

$\mathbf{x}$	vector of sectoral gross outputs
$\mathbf{f}$	vector of sectoral final demand categories
z_{ij}	intermediate flow from sector *i* to sector *j*
$\mathbf{A}$	direct input coefficients matrix with typical element a_{ij} $(= z_{ij}/x_j)$ indicating direct intermediate input requirements from sector *i* per unit of gross output of sector *j*
$\mathbf{I}$	identity matrix of appropriate dimension
$\mathbf{L}$	Leontief inverse matrix, i.e. $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$, with typical element l_{ij} indicating total (direct and indirect) intermediate input requirements from sector *i* per unit of final demand of sector *j*
$\mathbf{e}$	sectoral employment (or any other factor) figures
$\mathbf{M}$	employment (or any other factor) multiplier matrix, with typical element m_{ij} indicating total employment (or factor) generated in sector *i* per unit increase of final demand of sector *j*
m_j	employment (or any other factor) multiplier of sector *j*, i.e. $m_j = \sum_i m_{ij}$, indicating overall economy-wide increase in employment (or other factor) due to a unit increase in final demand of sector *j*
$\mathbf{V}$	Make matrix (industry by product)
$\mathbf{U}$	Use matrix (product by industry)
$\hat{\mathbf{r}}$	diagonal matrix with regional purchase coefficients along its diagonal
E	expectation operator

2. DETERMINISTIC ERROR ANALYSIS

In the early days when input-output (IO) analysis became one of the mainstream lines of economics research, computer speed and capacity posed problems for its implementation, especially since inversion of large-scale matrices was simply impossible.[4] Therefore, a part of the research focus was on the efficient use of the then

[4] On this issue, Miller and Blair (2009, p.31) write: 'In 1939 it reportedly took 56 hours to invert a 42-sector table (on Harvard's Mark II computer;…). In 1947, 48 hours were needed to invert a 38-sector input–output

state-of-the art technologies. Sherman and Morrison (1950) with the aim of 'eliminating the necessity of computing the new inverse from the beginning' (p.124), derived a closed-form formula for the changes in all entries of the inverse matrix due to a change in one element in the original matrix. Similar results were obtained in their earlier paper for changes occurring in the elements of a given column or row of the original matrix (Sherman and Morrison, 1949), which are a particular case of the Woodbury (1950) matrix identity that allows studying changes in elements in several columns or rows. Also Dwyer and Waugh (1953) presented methods of computing extreme bounds for discrepancies in the elements of the inverse matrix when errors in the original matrix are unknown but with specified limited bounds. In subsequent years, these results, especially the Sherman-Morrison formula, have been widely used in the IO literature, for example, in identifying the so-called 'important coefficients' defined as input coefficients having a particularly strong impact on the outcomes (e.g. sectoral gross outputs) or the elements of the Leontief inverse (also referred to as 'inverse-important' coefficients).[5]

Evans (1954) seems to be the first contributor to IO error analysis. The author questioned whether relatively small, but practically inevitable, errors in an input matrix (or 'structural matrix' in Evans' terminology) can cumulate to produce large and serious errors in the output estimates derived from the IO model. Using some of the mathematical results mentioned above, Evans first concludes that 'a very important property of input-output matrices' is the fact that an error in an input matrix, which is within the limits implied by the basic IO equation, 'will leave unchanged or introduce an error *of the same sign* in *every element* of the corresponding Leontief inverse' (p.464, emphasis added). Thus, errors opposite in sign will necessarily have *compensating effects* on the Leontief inverse entries, and importantly 'the effect of all structural errors taken together will be within the limits set by considering the positive and negative errors separately' (p.465). The author also analyses in detail: (i) the case of errors in final demand (or 'autonomous vector'); (ii) the impact of errors in a single row and two rows of an input matrix both theoretically and empirically, assuming identical percentage errors (of 5%) to obtain extreme results; (iii) the relations (equality and rough upper limit) between coefficient of variation (CV) of a sector's output and the CVs of input coefficients with errors (assumed in one row of an input matrix) when

matrix. However, by 1953 the same operation took only 45 minutes. … By 1969 a 100-sector matrix could be inverted in between 10 and 36 seconds, depending on the computer used.'

[5] For details and a large number of references on the concept of 'important coefficients' see, e.g., chapter 12.3 in Miller and Blair (2009).

considering the case of stochastic and independent input coefficients with constant CVs; (iv) errors in the activity level estimates when in place of ‘somewhat more suitable linear interrelationship functions’ (p.477) the typical IO proportionality assumptions are used; and (v) the impact on the Leontief inverse entries of the assumed identical and same-sign errors equal to a maximum bound in all the elements of the input matrix (a rather improbable case in practice, but sheds a great deal of light on the relative size of biases in the estimates due to errors). The main conclusions from these additional analyses reveals that (a) errors in input matrices have non-cumulative and compensating effects, (b) the error-compensating properties of the IO approach is further enhanced if the estimates of final demand, value-added and gross outputs are reliable (which implies that it is worthwhile to focus on the accuracy of these data, in particular), (c) the CVs of gross outputs are much less than those of input coefficients, and (d) the proportionality assumptions yield adequate results ‘over a fairly wide range of problems if used with reason and care’ (p.479).[6] Despite the fact that Evans (1954) was (one of) the first paper(s) focused on error-type analysis, it is, in our view, also a very important contribution to the topic in general for at least the reason that the author explores a wide variety of issues - relevant still today - in one study, which include deterministic and stochastic errors, dependent and independent errors, theory and empirics, and even uncertainty related to the IO modeling assumption of proportionality.

Independently of Evans (1954), it turns out that the IO property of partially compensating effect of errors in opposite directions on final estimates had also been revealed by Christ (1955, see footnote 22, p.151). Christ applies the earlier mentioned Dwyer and Waugh (1953) mathematical work on upper bound deviations in the inverse matrix entries due to errors in the original matrix to the 44-sector 1947 IO table of the US to assess the effect of ‘errors due to inaccurate data’. Further taking into account the fact that, unlike his assessment exercise, in reality errors in an input matrix will not be of the same algebraic sign, Christ concludes that he is ‘willing to hazard the generalization that the errors in the inverse caused by errors of given size in the input-output matrix are probably not as much as an order of magnitude larger than their parent errors in the input-output matrix, and that the resulting percentage errors in the predicted total outputs of the various industries are even less important’ (pp.157-158). In his review of IO analysis, Christ (1955) also in detail discusses the issue of errors

[6] See also Leontief (1955, p.18-19) on how IO strict proportionality assumptions can be refined to adequately capture nonlinear relationships.

due the IO modeling assumptions.[7] Here however, unlike Evans (1954), Christ's assessment is inconclusive and is based on his thorough discussions of theoretical and empirical evaluation of IO analysis. For example, the author states that the assumption of independence of inputs proportions from their relative prices in IO modeling violates 'accepted economic theory' (p.159).

However, not everyone felt comfortable with the statements on unimportance of errors due to the presence of offsetting errors in IO transactions matrix given that its row and column sums are known with high degree of confidence. Henderson (1955), for example, rightly observes that 'one error implies the existence of at least three others' (p.25) and then claims that the offsetting of errors works only with proportionate changes in final demand for all the industries whose inputs have been affected by the error. To justify his position, Henderson (1955) notes that every final demand vector in period 1, $\mathbf{f}_1$, can be decomposed as $\mathbf{f}_1 = \beta\mathbf{f}_0 + \mathbf{f}_\Delta$, where β is a scalar, $\mathbf{f}_0$ is the vector of final demand in period 0, and $\mathbf{f}_\Delta$ is the vector of free-in-sign values that sum up to zero, and thus re-writes the well-known Leontief model as

$$\mathbf{x}_1 = \beta\mathbf{x}_0 + (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}_\Delta, \qquad (1)$$

where $\mathbf{x}_1$, $\mathbf{x}_0$, $\mathbf{I}$ and $\mathbf{A}$ are the vectors of gross outputs in periods 1 and 0, identity matrix of appropriate size, and the input matrix, respectively. Henderson then claims that since $\mathbf{f}_\Delta$ sums to zero, the second term in (1) is not subject to the offsetting effects of balancing errors, as 'the overestimate of the inverse of one element will be accentuated by the underestimate of another element, if the two elements are to be multiplied by [the entries in $\mathbf{f}_\Delta$] of different sign' (p.25).

Park (1973) provides closed-form expressions for type I and type II output multipliers with the assumption of additive errors in IO coefficients (including errors in labor inputs and consumption coefficients), where two error components are separated from the 'true' multipliers. The first error component is associated with errors only in technical coefficients but not in the household vectors, while the second error component includes also the effects of errors in the household vectors (i.e. labor inputs and/or consumption coefficients) in addition to those in the technical coefficients. Further, Park shows that Type II output multipliers are a constant multiple (which is greater than 1) of the corresponding Type I output multipliers, and their errors are related by this constant

[7] In fact, Christ (1955) also discusses two additional types of errors: errors due to rounding and errors due to the use of approximation formulas. We ignore their discussion because with modern computer capacity

value as well.[8] Somewhat similar to Park (1973), a technique and its application to South Australian economy with the aim of discovering the effects of errors (represented by proportional changes of the original IO coefficients) in each individual input coefficient or set of coefficients on output, income and employment multipliers without the necessity of recalculating the last were discussed in West (1982). The author finds his method useful, in particular, for key sector identification problems, where also interval estimates of input coefficients can be translated into the corresponding interval estimates of the multipliers.

Sebald (1974), and Bullard and Sebald (1977) investigate the effects of uncertainty in input coefficients on the value of the so-called 'importance function' (which could have a scalar, vector or matrix dimension, e.g. defined as total output of a sector, employment by sectors, or Leontief inverse, respectively). They define an input coefficient as 'important parameter' if its perturbation causes the resulting changes in some elements of the importance function to exceed a pre-specified threshold, using the Sherman-Morrison (1950) relation. Further, they discuss norm bounds and worst-case bounds for solution tolerances due to uncertainty/errors in input coefficients, extending 'Christ's (1955) results to obtain tighter error bounds for a broader class of I-O models' (Bullard and Sebald, 1977, p.76). The worst case bounds are found by creating two perturbed input matrices such that one of them causes the greatest possible simultaneous increase or worst case positive tolerances on *all* elements of the Leontief inverse, while the second results in the worst case negative element-by-element tolerances on the Leontief inverse. Sebald (1974) explains how the corresponding two input matrices can be obtained. Using 101-sector US IO data, Bullard and Sebald (1977), for example, find that if all the entries in the input matrix are perturbed by ±10%, the resulting tolerance interval for total electricity demand is [-23.4%, 30.4%]. On the other hand, if the relevant 2% of the most important parameters remain fixed in the given exercise, the resulting tolerance interval significantly narrows to [-4.2%, 4.7%]. Similar results are obtained for a vector importance function. These results have important implications in terms of '1) setting priorities for data acquisition programs to update parameters, 2) constructing more accurate I-O tables for a given cost, and 3) identifying technologies where small changes have maximum impact on a policy objective, such as energy conservation' (p.80).

these errors are not relevant. In any case it turns out that even in those days (50's) these errors were of no importance for IO analysis.

[8] In fact, it was Sandoval (1967) who first proved that Type II *income* multipliers are a constant multiple of Type I income multipliers.

Sebald's (1974) notion of inverse important input coefficients was implicitly generalized to the inverse important *set* of input coefficients (that is, 'synergetic effects' with largest economy-wide impacts due to *simultaneous* changes of two, three, etc. elements of an input matrix in any desired order) by Sonis and Hewings (1989, 1992, 1995), who also introduce the notion of 'fields of influence'. Their analytical formulation of the difference between Leontief inverses with and without errors/changes in an input matrix (see Proposition 2 in Sonis and Hewings, 1992) is indeed a powerful result, and generalizes all closed-form expressions obtained by others for the mentioned difference with specific assumptions imposed on the input error structure matrix (e.g. changes in one element, one row and/or one column, two rows and/or two columns, and biproportional changes such as in the Generalized RAS bi-proportional updating procedure[9]). In the Sonis-Hewings formulation, more than one input coefficient taken in any order (not necessarily located in one row or column) can be changed to explore, for instance, the sensitivity of the Leontief inverse elements or any other importance function (in the terminology of Sebald, 1974 and Bullard and Sebald, 1977) due to certain synergetic effects. When the last are considered as errors, the mentioned contribution of Sonis and Hewings become immediately relevant for the IO uncertainty analysis.[10]

The *eigenvector approach* of Dietzenbacher (1990) is an appealing mathematical method that can be used to 'deduce straightforward *qualitative* results' (p.244, emphasis added) in terms of the impact of any type of error or perturbations in IO coefficients (e.g. additive errors, multiplicative errors, error rectangles, error couple, price effects, or more complex error structures) on IO multipliers. The main tool of this type of analysis is a lemma (Dietzenbacher, 1990, p.244) that allows indicating sectors whose multipliers are influenced the most and/or the least by an error structure chosen. However, all these results are stated in terms of inequalities, thus the author recognizes that 'a consequence of the nature of our expressions is that they may be of limited use in computational work' (p.241).

[9] For details about the (G)RAS updating procedure see e.g. Stone (1961) and Temurshoev, Miller and Bouwmeester (2013).

[10] See Temurshoev (2010, chapter 6.3) for the link between Sonis and Hewings' fields of influence approach and a measure of key group of sectors determined on the base of hypothetical extraction method.

3. ECONOMETRIC AND OTHER (NON-BAYESIAN) STATISTICAL APPROACHES

Attempts to account for uncertainties in IO data, in particular, those assuming random measurement errors, led to a large literature on statistical estimation of the parameters of the IO model as opposed to their mere computations. The very first studies attempted to estimate IO coefficients from the following set of *n* simultaneous equations

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f} + \boldsymbol{\varepsilon}\,, \tag{2}$$

i.e. a disturbance term $\boldsymbol{\varepsilon}$ is added to the open Leontief model. If long enough time series on **x** and **f** are available, input coefficients could be estimated from (2). Such *time series approach* has been pursued by e.g. Stone (1955), Briggs (1957) and Klein (1974, pp.341-342). In particular, Briggs (1957) discussed the issue of efficient estimation of IO coefficients focusing on ordinary least squares (OLS) and maximum likelihood estimation techniques.

However, Gerking (1976a) points to two problems with the time series approach. First, sufficiently long time series is required to estimate n^2 parameters in (2) by means of regression. Briggs (1957) had to implement his estimations at a high level of aggregation, because his time series consisted of only 6 observations (in fact, these were post-war British national accounts data used in Stone, 1955). Secondly, and of equal importance, Gerking noted that even with the availability of a long enough data series, 'such problems as structural change and variations in product mix may render these estimates meaningless' (Gerking, 1976a, p.276). The estimation of technical coefficients together with their standard errors from a set of cross-sectional data (i.e. *cross-sectional approach*) was first proposed by Gerking (1976a, 1976b). Primary data for building IO tables are obtained from representative firms/establishments in all sectors of an economy and, in general, can include information on firms' purchases only, sales only or both purchases and sales. Apparently, conducting the last type of survey provides more information, but is more expensive. The estimates of IO coefficients and flows derived from purchases (resp. sales) only data are referred to as 'columns only' (resp. 'rows only') estimates, in analogy to the structure of IO transactions table. Gerking (1976a) discusses the estimation procedure of 'columns only' estimates, while both types of estimates and their optimal combination in the derivation of 'reconciled' estimates are examined in Gerking (1976b).[11] In the case where firms' purchases and sales data are available and are to be used for building IO tables, the problem of reconciliation of 'columns only' and 'rows only' estimates arises, as in general the two sets of data will be inconsistent. Here we do not discuss this issue further, but briefly present Gerking's approach to estimation of IO coefficients for the 'columns only' case. The two-stage least squares (2SLS) estimators of the IO coefficients α_{ij} are derived from the following set of equations, which is estimated separately for each *j*-th column (i.e. purchasing sector *j*):

$$X_j(k) = \sum_i Z_{ij}(k) + W_j(k) + G_j(k) + R_j(k) \ , \tag{3.a}$$

$$Z_{ij}(k) = \alpha_{ij} X_j(k) + \theta_{ij}(k), \quad \text{for all } i = 1, \dots, n, \tag{3.b}$$

$$R_j(k) = \alpha_{n+1,j} X_j(k) + \theta_{n+1,j}(k) \ , \tag{3.c}$$

where $X_j(k)$, $W_j(k)$, $G_j(k)$, $R_j(k)$ and $Z_{ij}(k)$ are, respectively, establishment/firm *k*'s (from sector *j*) gross output, payments to households (wages and salaries), payments to government (e.g. property taxes), residual part of the value added including imports, and intermediate flows from all establishments in sector *i* to establishment *k* in sector *j*, while $\theta_{ij}(k)$ is an i.i.d. error with zero mean for all *k*. Thus, in the 2SLS approach the assumed error-free observations of $W_j(k)$ and $G_j(k)$ are used to construct an instrument for $X_j(k)$, resulting in consistent estimates of α_{ij}. Otherwise, 'if both $X_j(k)$ and $Z_{ij}(k)$ are measured with error …, the OLS estimate of α_{ij} will be both biased and inconsistent' (Gerking, 1976a, p.277).[12] Note that equation (3.b) implies a strong assumption of all firms in each sector having identical production functions of a Leontief type. It should also be noted that Miernyk (1976) and Brown and Giarratani (1979) raised some criticism with regard to Gerking's approaches; subsequently, he incorporated those that he recognized as missing ones (namely, purchases and sales identities and a priori judgmental information) in Gerking (1979a).[13] Nevertheless, further disagreement between Miernyk and Gerking, in particular, on the appropriateness of deterministic vs. stochastic IO models, resulted in additional

[11] See also Gerking and Pleeter (1977) who based on Gerking (1976b) outline a method for choosing sample sizes for IO studies in order to build minimum variance IO tables of estimated regional coefficients and minimum variance output forecasts.

[12] In fact, due to this possible problem, Gerking (1976a) also used two other instrumental variable techniques of the Wald-Bartlett method (Wald, 1940; Bartlett, 1949) and the Durbin's method (Durbin, 1954). Gerking's results indicate that these two methods are less efficient (i.e. they resulted in higher standard errors of α_{ij}) than 2SLS.

[13] See also Gerking (1979b).

exchanges, in which Miernyk essentially supported only survey-based methods and professional judgement, while Gerking focused on the importance of statistical estimation approaches (see Miernyk, 1979; Gerking, 1979c). Finally, Hanseman and Gustafson (1981) proposed 'a reinterpretation and simplification of Gerking's stochastic input-output model' (p.470), the details of which are not given here due to space constraints.

Finally, intrasectoral heterogeneity are modeled in terms randomly varying input (or output) coefficients in Kockläuner (1989). For example, the structural equation of the purchase model, equivalent in interpretation to the 'column-only' setting of Gerking (1976a), is defined as

$$Z_{ij}(k) = \alpha_{ij}(k)X_j(k) + \theta_{ij}(k), \quad \text{for all } i = 1, \dots, n, \tag{4.a}$$

which differs from (3.b) in that now the input coefficient is specific for each establishment *k*, and 'is assumed to vary randomly around a sectoral mean' (Kockläuner, 1989, p.312), i.e.

$$\alpha_{ij}(k) = \alpha_{ij} + \omega_{ij}(k), \tag{4.b}$$

where $\omega_{ij}(k)$ is a stochastic disturbance with zero mean. The author discusses how to estimate such regressions, including the relevant macro-equations, 'perfect aggregation structure' (p.313), reconciled estimation of the purchase and sale models, and the underlying properties in finite populations.

Unlike Gerking's focus on individual input coefficients estimation, Braschler (1972) uses OLS regression analysis to estimate sector-specific multipliers, where the relevant single equation model is derived from the IO model. The author, however, does not mention accounting for uncertainty with this approach, but rather sees it as an alternative option with costs of implementation being minimal in comparison to the IO approach that requires primary data. Consider the open Leontief model of $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ = $\mathbf{Lf}$, where $\mathbf{L}$ is the Leontief inverse matrix. If we pre-multiply this equation by the direct employment coefficients sector-wise, we obtain **e = Mf**, where **e** is the sectoral employment vector, and **M** is the employment multiplier matrix. Summing over all sectors, the relationship between total employment and sectoral final demands can be written as $\sum_i e_i = \sum_j \sum_k m_{kj} f_j = \sum_j \beta_j f_j$, where $\beta_j \equiv \sum_k m_{kj}$ is the employment multiplier of sector *j*. Therefore, the single equation model that Braschler used in his estimation of sectoral employment multipliers for Missouri, has the following regression form:

$$Y = \beta_0 + \beta_1 E_1 + \beta_2 E_2 + \cdots + \beta_n E_n + \varepsilon \,, \tag{5}$$

where Y is total employment (or gross output, or any other *economy-wide* variable of interest), E_i is final demand of industry *i*, and ε is the disturbance term with usual properties of classical linear model. Braschler (1972) uses county observations as cross-sectional data for the empirical estimation of (5), where exports from the local economies are taken as the explanatory variables, E_i's. This choice of regressors is made because regression (5) would be appropriate if the 'final demand vector is assumed to be completely exogenous to the local or regional economy containing the endogenous local industries' (p.459). This justification is related to the basic (export) vs. nonbasic (service or local) activities/industries distinctions in related literature of *economic base studies* that also run regressions somewhat similar to (5) with service employment as dependent variable in order to estimate regional employment multipliers (see e.g. Hildebrand and Mace, 1950; Weiss and Gooding, 1968; Park, 1970).

Dealing with a different issue of the problem of negative IO coefficients also led to consideration of the issue of the *sensitivity* of IO coefficients to the errors in the underlying data in supply and use tables (SUTs). Ten Raa and van der Ploeg (1989) presented a maximum likelihood methodology in order to solve the problem of negative IO coefficients arising from adopting a 'theoretically superior' commodity technology model. In obtaining new, adjusted SUTs data such that the underlying input matrix does not contain negatives, the estimates of variances of the original SUTs elements are required. Although on the base of their results applied to the UK data, the authors rejected the commodity technology model, their approach in accounting for errors in SUTs data, including sensitivity of input coefficients to SUTs entries, is potentially useful for dealing with other related issues, e.g. sensitivity of Leontief inverse elements due to the errors in SUTs. Similar topics are discussed, among others, by ten Raa (1988), Mattey and ten Raa (1997), and Rueda-Cantuche and ten Raa (2013).

Since IO tables are analytical constructs of SUTs, it seems reasonable to use directly SUTs for the analyses of various IO-related issues, if such choice turns out to be feasible. Ten Raa and Rueda-Cantuche (2007) were the first to realize that, for the case of the so-called product technology assumption, it is possible to obtain the estimates of products/commodities IO multipliers (and their confidence intervals) directly from *establishment SUTs* – supply and make tables at the level of establishments – by running an appropriate econometric regression. Exactly the same methodology has been applied to the widely available economy-wide SUTs data at the industry level in Rueda-Cantuche and Amores (2010) and Rueda-Cantuche (2011). The last study refers to the method as the supply-use based econometric (SUBE) approach. Ten Raa and Rueda-Cantuche (2007) show that under the product technology assumption, that states that each product is produced in its own specific way irrespective of the industry in which it is produced, the commodity-specific generalized (e.g. employment, pollutant emissions) IO multipliers, $\boldsymbol{\beta}$, can be estimated by running the following regression:[14]

$$\mathbf{e} = (\mathbf{V} - \mathbf{U}')\boldsymbol{\beta} + \boldsymbol{\varepsilon}\,, \tag{6}$$

where **V** and **U** are, respectively, make and use matrices, thus each column of $(\mathbf{V} - \mathbf{U}')$ represents *net outputs* per product across all establishments (or industries), **e** is the vector of employment figures (or any other variable of interest), and $\boldsymbol{\varepsilon}$ is a vector of independent and normal random disturbance errors with zero mean and constant variance. Of course, running (6) makes sense only with rectangular SUTs when the number of observations (i.e. establishments) is larger than that of commodities. Thus, for all kinds of IO 'multipliers, the huge sample size justifies our normality assumption, by the Central Limit Theorem' (ten Raa and Rueda-Cantuche, 2007, p.322). When instead of establishments SUTs, economy-wide SUTs that in practice usually distinguish more products than industries, are used, it is necessary that the relevant product dimension is reduced in order to have sufficient degrees of freedom in implementing (6). In their SUBE application using economy-wide SUTs, Rueda-Cantuche and Amores (2010) 'assume that the error term only includes *sampling errors* by assuming linearity, a correct specification of the model without relevant omitted variables, and that the randomness of human behaviour and the measurement errors do not significantly affect the results' (p.989, emphasis added). Finally, Rodrigues and Rueda-Cantuche (2013) extend the SUBE approach by explicitly taking into account the restriction that at the overall system-wide level the implied total factor (e.g. emissions) requirements matches the corresponding observed values.

[14] See Temurshoev (2012, Table 2, p.11) for the respective regression-form equations in terms of SUTs for other widely-used transformation models (i.e. those based on industry technology, fixed industry sales structure, and fixed product sales structure assumptions).

4. RANDOM ERROR ANALYSIS AND PROBABILISTIC APPROACH

Another approach towards quantification of uncertainties in IO modelling, arguably better fitted for uncertainty analysis compared to the deterministic error analysis, is random error analysis or probabilistic approach, in which IO model components (e.g. input coefficients or transactions) are assumed to be stochastic (equivalently, random) variables. This implies that input coefficients (and possibly other components of the IO table) are distributed according to some probability distribution.

Quandt (1958) justifies his treatment of input coefficients as stochastic variables on the basis of a 'structural rationale' and a 'sampling rationale'. The reasoning behind the first rationale lies in the belief that quantity demanded changes in a probabilistic manner, which implies that observed IO coefficients are also (under certain conditions valid within the IO setting) random variables. The sampling rationale, on the other side, means that the value of an input coefficient could be obtained 'by taking a random sample of individual firms stratified according to the desired industry classification. The sample mean for each coefficient would provide an estimate of the derived coefficient' (p.156). For simplicity sake, Quandt (1958) assumes that input coefficients are independently distributed random variables, claiming that the condition for the existence of the Leontief inverse (i.e. the sum of input coefficients being less than unity for all sectors) does not necessarily imply dependence, while '[t]he distributions of coefficients in a particular column may be such that even if all coefficients assume their extreme values, their sum will still be less than unity' (p.156). Quandt then assumes that some 'true' input coefficients matrix exists and therefore compares the 'true' Leontief system with the probabilistic one. He proves an important theorem stating that 'the expectation of the solution of a random Leontief system is not necessarily equal to the solution of the true system' (p.158). Next, Quandt investigates the issue of determination of variances of gross outputs as a solution of the Leontief system in order to establish confidence intervals for the solution that obviously depends on the knowledge of variances and covariances (along the row) of the elements of the Leontief inverse.[15] For this purpose, he first establishes general approximating formulae for (co)variances of fractions, on the base of which closed-form expressions for the (co)variances of the Leontief inverse elements in a simplified two-sector economy under the assumption of independent and symmetrically distributed errors with zero means are derived. To obtain the confidence intervals for gross outputs, an additional assumption on (or an empirical knowledge of) the solution distribution is needed. Three such intervals for the two-sector example are considered and compared, including intervals based on normally distributed errors (using the approach of Box and Hunter, 1954), Quandt's approximation formulae and the critical values of the normal distribution, and Chebyshev's inequality.

Instead of gross outputs, McCamley, Schreiner and Muncrief (MSM, 1973) focus on the estimation of the *reliabilities* (sampling variances) of employment multipliers on the basis of a sampling variance matrix for the IO transactions table estimators. The existence of the transactions table covariance matrix is justified by the sampling rationale, similar to Quandt's (1958) rationalization of stochastic input coefficients. That is, primary data for the IO transactions table come from a sample of firms in each sector, and if the data gathering procedure could have been repeated, 'a different sample of firms and thus a somewhat different transactions table would have been obtained' (MSM, 1973, p.83). Using the well-known (in econometrics literature) Delta method, the authors derive the following approximating formula for the variance of employment (or any other factor) multiplier of sector *k*:

$$\mathrm{Var}(m_k) \approx \sum_{i,j,q,r} \frac{m_i l_{jk}}{x_j} \mathrm{Var}(z_{ij}, z_{qr}) \frac{m_q l_{rk}}{x_r}, \tag{7}$$

where m_k is employment multiplier of sector *k*, l_{jk} is the *jk*-th element of the Leontief inverse, x_j is gross output of sector *j*, z_{ij} is the interindustry transaction flowing from sector *i* to sector *j*, and $\mathrm{Var}(z_{ij}, z_{qr})$ is the covariance between the indicated flows (we use different notation). The covariance matrix estimate is obtained from the authors' data derived from interviewed firms' own employment and *sales* (in percentage terms) to other sectors. The nature of the data available allows for the use of a particular, more simplified computationally, version of (7), where the elements of the transactions covariance matrix are non-zero only when $i = q$.[16] MSM (1973, section 4.2) discusses the details of the approach used in deriving the transactions covariance matrix in their empirical application of the proposed method.

To account for randomness in input coefficients and final demand, Goicoechea and Hansen (1978) replace the *i*-th equation from the deterministic open Leontief model with a probability statement stating that 'the number of times (expressed as a

[15] The moments of fractions approach is used because each entry of the inverse matrix is a fraction with its numerator and denominator being a cofactor and determinant, respectively.

percentage) interindustry use and final consumption use are less than or equal to the output of sector *i* is $1 - \alpha_i$' (p.286), where $\alpha_i \in [0,1]$. Choosing any type of probability density function for input coefficients and final consumption is possible in this setting, whose parameters need to be defined such that e.g. the expected values of these densities equal the corresponding observed values. The probabilistic statements for all industries then are transformed 'into an equivalent system of nonlinear but deterministic equations, with an uncertainty level already built into [them]' (p.288). As an example, the authors choose (independent) exponential distributions to represent the randomness in input coefficients and final consumption (considering a 2-sector economy). One of their interesting findings is that the output levels obtained from the deterministic approach are achievable only with 60% probability 'if in reality the [input coefficients and final consumption] follow an exponential p.d.f.'. The approach naturally accommodates sensitivity/error analysis of the density parameters of the input coefficients and/or final demand categories.

Many papers have investigated the issue of underestimation or overestimation of the 'true' Leontief inverse in practice. Assume that IO coefficients in **A** are random variables, then $E\{(\mathbf{I}-\mathbf{A})^{-1}\}$, where E is expectation operator, is the *practical estimate* of the *true* Leontief inverse that is defined as $(\mathbf{I}-E\{\mathbf{A}\})^{-1}$, where $E\{\mathbf{A}\}$ is the true IO coefficients matrix.[17] Simonovits (1975) proved that if all coefficients of **A** are independent random variables, then

$$E\{(\mathbf{I}-\mathbf{A})^{-1}\} \geq (\mathbf{I}-E\{\mathbf{A}\})^{-1}, \tag{8}$$

i.e. *with independent IO coefficients, in practice the Leontief inverse is overestimated*. This result implies that with a fixed final demand vector, the practical estimator of the gross outputs vector will be overestimated as well. Simonovits (1975) further showed

[16] In case each firm supplies information about its purchases rather than sales, the transactions variance matrix in (7) will have non-zero elements only when $j = r$ (MSM, 1973, p.88).

[17] We should note that these definitions are reversed in Simonovits (1975). In fact, in the literature, particularly, in the relevant earlier contributions, there is confusion regarding under- or over-estimation of the Leontief inverse in practice. This confusion has been discussed and 'solved' in Lahiri and Satchell (1986). The reasoning of this confusion is as follows. Input matrix is first written as $\boldsymbol{A} = \bar{\boldsymbol{A}} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is the matrix of random variables with zero means, while $\bar{\boldsymbol{A}}$ is the deterministic input coefficients matrix. Among others, Simonovits (1975), Lahiri (1983), and Flåm and Thorlund-Petersen (1985) considered $E\{\boldsymbol{A}\} = \bar{\boldsymbol{A}}$ as the observed IO coefficients matrix, and ***A*** as its true (and unknown) counterpart. This would imply that the observed IO coefficients are deterministic, while the true IO coefficients are stochastic. 'However, since the major source of randomness in the IO coefficients is errors in measurement or observation, it seems rather odd to assume that the observed IO coefficients are deterministic' (Lahiri and Satchell, 1986, p.71). Thus, instead it makes more sense to define the entries of $E\{\boldsymbol{A}\} = \bar{\boldsymbol{A}}$ as the true IO coefficients that are deterministic, and those of ***A*** as their observed counterparts that are subject to random errors.

that when **A** has a *single error rectangle*[18] (hence, four IO coefficients are dependent) and the row and column sums are fixed, there are at least two elements of the Leontief inverse one of which is overestimated and the other is underestimated. Note that the last finding is related to Evans' (1954) and Christ's (1955) assertions of compensating error effects, discussed in Section 2.

Relaxing Simonovits' (1975) assumption of independent IO coefficients to a particular class of dependent IO coefficients, Lahiri (1983) finds that the inequality (8) is still valid. This particular class of distribution is the assumption of *biproportionally stochastic* IO coefficients that are defined as $a_{ij} = r_i \tilde{a}_{ij} s_j$ for all sectors *i* and *j*, where $\tilde{a}_{ij}$'s are deterministic and r_i's and s_j's are independently distributed random variables. This stochasticity definition of IO coefficients is entirely akin to the RAS updating technique, where the row multipliers r_i's and the column multipliers s_j's are claimed to capture the so-called substitution and fabrication effects, respectively (see e.g. Leontief, 1941; Stone, 1961; Bacharach, 1970). Additionally, Lahiri (1983) proves that in a non-linear IO system the gross outputs vector is underestimated (rather than overestimated) when the final demand vector is stochastic.

The results of independent input coefficients in Simonovits (1975) and of biproportionally stochastic coefficients in Lahiri (1983) that lead to practical overestimation of the Leontief inverse and gross outputs turn out to be particular cases of a more general condition of *moment-associated* random variables as put forward by Flåm and Thorlund-Petersen (1985). Non-negative random variables $z_1, z_2, \ldots, z_n$ with finite means are called moment-associated if for all non-negative integers $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ the following condition holds:

$$E\left(z_1^{\vartheta_1} \cdot z_2^{\vartheta_2} \cdot \ldots \cdot z_n^{\vartheta_n}\right) \geq (Ez_1)^{\vartheta_1} (Ez_2)^{\vartheta_2} \ldots (Ez_2)^{\vartheta_2}. \tag{9}$$

Obviously when variables are independent or identically distributed, inequality (9) holds. However, moment-association also implies non-negative covariance between any pair of variables when (9) holds for all $\vartheta_1 + \vartheta_2 + \ldots + \vartheta_n = 2$. Applying (9) to the input matrix and final demand vector, Flåm and Thorlund-Petersen (1985) prove that the practical estimator of the Leontief inverse is overestimated if input coefficients are assumed to be non-negative moment-associated random variables, which imply that for any random input matrix **A** the inequality $E\{\mathbf{A}^k\} \geq \{E\mathbf{A}\}^k$ is valid for all $k = 0,1,2, \ldots$

[18] Error rectangle as simplest case of more general family of distributions has been widely used in IO error analysis (see Bródy, 1970, p.128).

.Similarly, they find that the vector of gross outputs is overestimated in practice under the assumption that all input coefficients and final demands are non-negative moment-associated random variables, implying that for $k = 0,1,2, \dots$ the inequality $E\{\mathbf{A}^k\mathbf{f}\} \geq \{E\mathbf{A}\}^k\{E\mathbf{f}\}$ holds for any random input matrix **A** and any random final demand **f**.

Lahiri and Satchell (1986) show that the results of Simonovits (1975) and Lahiri (1983) also hold under the assumption of non-negative expected value of the overall product of all random IO errors.[19] Finally, the single error rectangle assumption is generalized to a more reasonable, from a practical point of view, assumption of an arbitrary *series of error rectangles* by Dietzenbacher (1988), who showed that each nonzero row and column of the Leontief elements *bias* matrix, defined as the difference between the observed (stochastic) Leontief matrix and the true (deterministic) Leontief matrix, must contain entries with opposite signs. The author further proves the same result for the difference matrix of the observed and the 'true' integrated/extended coefficients, where errors in the direct factor (value-added) coefficients are also allowed. Dietzenbacher (1988) further considers the case when instead of the usual assumption of fixed row and column sums of the complete IO flow matrix, only the condition of consistency of total inputs and total outputs is imposed. In this case, errors may occur as a couple (hence, the term *error couple* is introduced in the paper) in addition to the rectangle. In the error couple setting, similar results were obtained only for the rows of the Leontief bias matrix. To conclude, all these findings essentially further confirm the early results on error compensating effects, in particular with regard to uncertainties in aggregate outcomes such as multipliers or gross outputs.

Randomness in input coefficients could be introduced due to uncertainties (or errors) in estimated prices of aggregate commodities. The obtained bias, either overestimation or underestimation, in the Leontief inverse due to errors in prices is discussed in Lahiri and Satchell (1985). They also consider another specification of input coefficients stochasticity, called *uniform errors* that are defined as (a) all entries of **A** being uniformly affected by a *single* random variable and/or (b) this single random variable has uniform distribution. For space consideration reasons, we do not discuss case-specific and rather extensive results of Lahiri and Satchell (1985) further here. The

[19] In addition, Theorem 6 in Lahiri and Satchell (1986) stated that it does not make a difference for under/over-estimation outcomes of the Leontief elements whether the random single error rectangularity assumption is stated in terms of IO coefficients, like in Simonovits (1975), or in terms of intermediate flows. However, Dietzenbacher (1995) proved this statement to be incorrect (p.383-384 and Appendix B).

main conclusion derived from this study is that the results are indeed mixed with both over- or under-estimation possibilities in the Leontief inverse entries.

West (1986) assumes IO coefficients errors to be normally and independently distributed with zero mean and variance σ_{ij}^2, and derives the following *closed-form* approximating expressions for sector *k*'s multiplier bias and variance, respectively:

$$E(\widetilde{m}_k) - m_k \approx \sum_{i,j}\left(m_i l_{ji} l_{jk} \sigma_{ij}^2\right)\left[1 - 7 l_{ji}^2 \sigma_{ij}^2\right]^{-3/7}, \quad (10)$$

$$Var(\widetilde{m}_k) \approx \sum_{i,j}\left(m_i l_{jk} \sigma_{ij}\right)^2 \left[1 + \tfrac{59}{16}\left(l_{ji}\sigma_{ij}\right)^2\right]^{128/59}, \quad (11)$$

where the tilde refers to stochastic variable, while the remaining terms are already defined right below (7). The main finding of West (1986) is that the expected value of the error term is positive (hence, the observed multipliers are *underestimated*) and the multiplier distribution is positively skewed. In his empirical application, West uses his approximating formulae and estimates of IO coefficients standard errors to analyse the main features of the output, income and employment multipliers of the Central Queensland economy. However, ten Raa and Steel (1994) criticized West's (1986) approach showing that his 'formulas do not hold for multi-sector economies and his stochastic assumptions admit no mean or variance, not even for single-sector economies' (ten Raa and Steel, 1994, p.365). Instead they adopt an alternative stochastic assumption for IO coefficients, namely independent beta distribution defined on the unit interval, which solves the moments' non-existence problem. Further, to avoid inconsistencies in assumptions in deriving multiplier density functions in a multi-sectoral setting, ten Raa and Steel directly evaluate the first and second moments of IO multipliers through Monte Carlo calculations using West's highly aggregated (5-sector) IO data. Surprisingly, they also find that West's (1986) approximating formulas result in values very close to their Monte Carlo counterparts that are based on different stochastic assumption. The reason for this similarity is that 'West assumes small variances. Under this assumption, the leading terms of his formulas can be shown to be first-order approximations to the mean and the variance of the Leontief inverse' (ten Raa and Steel, 1994, p.370; see also below).

Kop Jansen (1994), referring to the results of his earlier (unavailable to us) work in Kop Jansen (1992), presents (maximum lower bounds) approximating formulas for the first- and second-order moments of the Leontief inverse under the assumption of independently distributed IO coefficients (see equations (7) and (8) in Kop Jansen, 1994). Consistent with the ten Raa and Steel (1994) finding, Kop Jansen shows that his formulas for IO multipliers bias and variance equal West's (1986) approximations in first order (see below). Thereafter, Kop Jansen (1994) runs Monte Carlo simulations with independent (uniformly distributed) and dependent (error rectangles) IO coefficients and compares the obtained results in terms of output and income multipliers with those based on his and West's approximating formulas. On the basis of these comparisons, Kop Jansen recommends using (a) his formula as an approximation of the expected value of multipliers whether the variances of IO coefficients are small or large, and (b) West's (1986) multiplier variance formula (resp. Kop Jansen's multiplier variance formula) if the variances of IO coefficients are relatively large (resp. small).

It turns out that the assumption of *independent* random input coefficients implies an *inconsistent* underlying IO transactions matrix. This was shown by Dietzenbacher (1995), who proves that the assumption of independently distributed and unbiased intermediate and primary inputs coefficients and final demand vector imply that the underlying intermediate and primary inputs transactions table entries exhibit only positive biases in their row sums. Therefore, also considering the practice of constructing input matrices, Dietzenbacher (1995) suggests to 'impose the stochastics at the starting point of the analysis; that is, on the transactions table instead of on the coefficients matrices' (p.381). The author further reveals that when the transactions table is the source of random errors, the biases (either zero, or positive and negative) within each row of the multiplier matrix cancel each other out in the sense that their *weighted average* is zero, where the weights are known final demand totals that are the same for each row. Similar results hold for each column of the Leontief inverse. These analytical results are based on the condition that certain margins of the transactions table are known. Next, Dietzenbacher (1995) shows that 'the conjecture of unbiased multiplier estimates is not true in general' (p.385-386).

Ten Raa and Kop Jansen (1998) is an important contribution to stochastic IO analysis as they present the analytical first-order approximation formulas of the bias and sensitivity (or covariance) of the Leontief inverse entries (hence, IO multipliers as well) in a generalized setting of *dependent* (thus, also independent) stochastic IO coefficients. The authors show that the bias and sensitivity of interest are respectively approximated by:

$$E(\tilde{l}_{hq}) - l_{hq} \approx \sum_{i,j,k,p} l_{hi} l_{jk} l_{pq} \cdot \mathrm{Var}(\tilde{a}_{ij}, \tilde{a}_{kp}) \,, \qquad (12)$$

$$Var(\tilde{l}_{hk},\tilde{l}_{ps}) \approx \sum_{i,j,q,r} l_{hi}l_{jk} \cdot \mathrm{Var}(\tilde{a}_{ij},\tilde{a}_{qr}) \cdot l_{pq}l_{rs} \,, \tag{13}$$

where $\mathrm{Var}(\tilde{a}_{ij},\tilde{a}_{qr})$ is the covariance between the two indicated random IO coefficients (tilde refers to stochastic variable). Indeed, the importance of analytical expression (12)-(13) lies in the fact that they allow for dependent IO coefficients, while independence, discussed widely in the literature, is only a particular case captured by the formulae as well. That is, with independent IO coefficients, we have $\mathrm{Var}(\tilde{a}_{ij},\tilde{a}_{qr}) = 0$ for $(i,j) \neq (q,r)$ and $\mathrm{Var}(\tilde{a}_{ij},\tilde{a}_{ij}) = \sigma_{ij}^2$, and thus the bias and sensitivity in (12) and (13) simplify to, respectively:

$$E(\tilde{l}_{hq}) - l_{hq} \approx \sum_{i,j} l_{hi}l_{ji}l_{jq}\sigma_{ij}^2 \,, \tag{14}$$

$$Var(\tilde{l}_{hk},\tilde{l}_{ps}) \approx \sum_{i,j} l_{hi}l_{jk} \cdot \sigma_{ij}^2 \cdot l_{pi}l_{js} \,. \tag{15}$$

From (15), the own variances boil down to:

$$Var(\tilde{l}_{hk},\tilde{l}_{hk}) \approx \sum_{i,j}(l_{hi}l_{jk}\sigma_{ij})^2 \,. \tag{16}$$

Since the stochastic output multiplier of sector *k* equals $\tilde{m}_k = \sum_h \tilde{l}_{hk}$, summing over *h* in equations (14) and (16) yields the following bias and sensitivity of IO multipliers in case of independent IO coefficients:

$$E(\tilde{m}_k) - m_k \approx \sum_{i,j} m_i l_{ji} l_{jk}\sigma_{ij}^2 \,, \tag{17}$$

$$Var(\tilde{m}_k) \approx \sum_{i,j}(m_i l_{jk}\sigma_{ij})^2 \,. \tag{18}$$

Thus, (17) and (18) coincide with the leading terms of West's (1986) formulae given, respectively, in (10) and (11). These also match Kop Jansen's (1994) maximal lower bounds in the presence of independent IO coefficients. By the assumption of small variances, the terms in square brackets in (10) and (11) become very small and can be ignored. Thus West's formulae match first-order estimates of the 'true' bias and variance with independent IO coefficients.

Note that it is interesting to compare the approximating formulae of the variance of IO multipliers discovered by McCamley, Schreiner and Muncrief (MSM, 1973), and ten Raa and Kop Jansen (1998), respectively, given in (7) and (13). The first study considers a generalized setting with 'stochastic' IO transactions, while the second deals with stochastic IO coefficients. From equation (13) the variance of multipliers with *dependent* IO coefficients can be obtained as:

$$Var(\widetilde{m}_k) \approx \sum_{i,j,q,r} m_i l_{jk} \cdot \mathrm{Var}(\tilde{a}_{ij}, \tilde{a}_{qr}) \cdot m_q l_{rk} \,. \tag{19}$$

Assuming that total outputs are known (thus, deterministic) and recalling that IO transactions were denoted by z_{ij}, from covariance property we obtain $\mathrm{Var}(\tilde{a}_{ij}, \tilde{a}_{qr}) = \mathrm{Var}\left(\frac{\tilde{z}_{ij}}{x_j}, \frac{\tilde{z}_{qr}}{x_r}\right) = \frac{1}{x_j}\frac{1}{x_r}\mathrm{Var}(\tilde{z}_{ij}, \tilde{z}_{qr})$. The last if plugged back into (19) gives *exactly* the MSM's (1973) approximating formula in (7). To our knowledge, this relation has never been noticed in the literature so far.

Finally, while doing our best in going through all the literature dealing with the IO uncertainty problems, we made an important and, at the same time, surprising discovery. It turns out that already in the early 1980s, '[i]n modeling the interactive environment of Space Station, NASA engineers at Johnson Space Center (JSC) and system analysts at Jet Propulsion Laboratory (JPL) developed an approach that is similar in many respects to a Leontief input-output system. A distinctive feature of the model, however, is that the entries in the input-output matrix (the "housekeeping" matrix) are random variables' (Quirk, Olson, Habib-Agahi and Fox, 1989, p.585; for short QOHF). There are also other sources of uncertainty in their model, e.g. related to the so-called capital coefficients as well as cost parameters for the design and maintenance of Space Station. Now consider the usual open Leontief model, and denote $\mathbf{K} = \mathbf{I} - \mathbf{A}$ such that its inverse is equal to the Leontief inverse matrix, i.e. $\mathbf{K}^{-1} = \mathbf{L}$. Since the entries in $\mathbf{A}$ are random variables so are the elements in $\mathbf{K}$. Further, let $f(\mathbf{K})$ denote the *pdf* over $\mathbf{K}$, i.e. $f(\mathbf{K}) = f(k_{11}, \ldots, k_{nn})$ where k_{ij} is the typical element of $\mathbf{K}$ and *n* is the number of sectors. And, let $g(\mathbf{L})$ denote the *pdf* over the Leontief inverse matrix, given $f(\mathbf{K})$. QOHF (1989, pp.586-587) state that Fox and Quirk (1985; we were unable to find this work) assuming that the probability that the determinants of $\mathbf{K}$ or $\mathbf{L}$ are nonzero (except on a set of measure zero), derived the *pdf* of the Leontief inverse as having the following general form:

$$f(\mathbf{L}) = |\mathbf{L}|^{-2n} f(\mathbf{L}^{-1}) \,, \tag{20}$$

where $|\mathbf{L}|$ is the determinant of the Leontief inverse.

It is immediately clear that equation (20) is a very important finding/contribution as far as uncertainty of the multiplier matrix, hence the IO system in general, is concerned. It is a multivariate distribution, and changes its form depending on the assumed specifications of the density of the input coefficients matrix. Note also that (20) is general enough to allow for any sort of dependencies between input coefficients.

Details regarding various specifications of the probability density of the input matrix, the corresponding densities of the Leontief matrix, and uncertainties of the IO multipliers and solution are discussed in Temurshoev (2015b), which also provides the (most probably alternative) proof of (20).

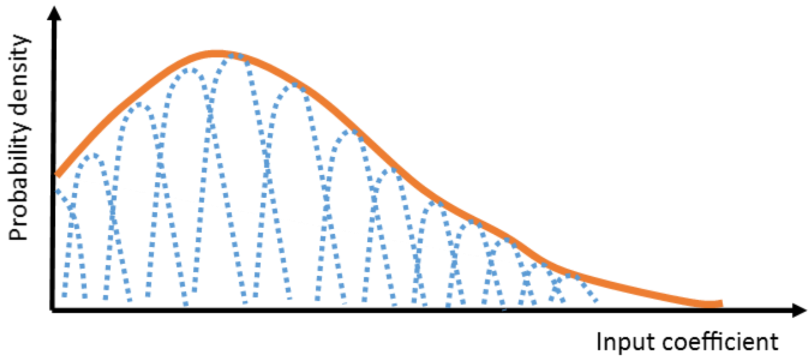
Coming back to our assertion that our discovery of the Fox and Quirk (1985) work through QOHF (1989) was a surprising finding: it is indeed surprising because it turns out that the entire IO community and other researchers that were actively involved in stochastic IO analysis to date were simply unaware of the Fox and Quirk contribution. To say the least, it *could* have eased the investigations regarding multiplier matrix uncertainty enormously. The most probable explanations for this unawareness, we think, have to do with the facts that: (a) economists, in particular IO economists, never even thought or think that there could be any direct relation between IO analysis and Space Sciences, and (b) compared to 1980's and 1990's, nowadays we abundantly reap the fruits of the development of modern technologies, like the internet, powerful search engines, and automatic research citations indications.

5. FULL PROBABILITY DENSITY FUNCTION (FULL PDF) APPROACH

In our view, an important contribution by Jackson (1986) has not been given due attention in the literature of stochastic IO analysis to date. Jackson (1986) refers to his method as 'full *pdf* approach to macrovariable representation' (*pdf* stands for probability density function), and distinguishes it from related earlier literature in that the full-distribution approach 'derives not from measurement and sampling error, but from an expected, systematic variation' (p.517) that is 'directly attributable to industrial, institutional, and location factors' (p.515). Further, 'the approach does not support the idea of random IO tables, [and] is specifically designed to augment the informational content of the macrovariable and of the macromodeling framework by more fully utilizing the microlevel component data' (p.529). Therefore, the fact that the traditional average production coefficients used in IO modelling reflect the diversity of commodities and processes, and diversities due to spatially variant location factors, ownership patterns and institutional influence is explicitly spelled out in this approach. For example, variables such as 'market structure, establishment size, prices, labor productivity, technology, product mix, and ages of capital stock, all of which vary regionally, are determinants of probabilistic structure of interindustry interaction' (Jackson, 1989, p.87). This implies, for example, that every input coefficient of an individual firm has a *pdf* reflecting all possible values of the coefficient considering the impact of all the factors mentioned, and not just observed input coefficients. Given that the full-distribution approach specifies the possible ranges of each input coefficient and associated probabilities, the resulting distributions of outputs (e.g. multipliers, estimates of employment, etc.) will also represent the probable ranges of the outputs and their associated probabilities. This is different, according to Jackson (1986), from error intervals due to input coefficients estimation, because 'even in the absence of measurement and sampling error and changes due to production technologies, one should expect a range of probable outcomes' (p. 517).

In the full *pdf* approach, the aggregate variable representation becomes more flexible, and at the same time measurement and sampling errors become less relevant as all possible firm input coefficients are represented. The reason is that while estimation of individual firm input coefficient *pdf*'s is subject to error, no such error would dominate the aggregate *pdf*, as the last encompasses/envelopes all relevant firms' coefficient *pdf*'s. Figure 1 illustrates an example of a relationship between individual firm input coefficient *pdf*'s, represented by the blue, dashed lines, and the appropriate aggregate level *pdf*, illustrated by bold orange, solid line. Obviously, the aggregate *pdf* is a weighted average of the individual firm *pdf*'s such as that it integrates to unity (i.e. it becomes a proper density function). In terms of aggregation issues, finer (resp. lesser) disaggregation of industries and/or regions would result in more kurtose (resp. flatter) aggregate distributions, reflecting smaller (resp. greater) uncertainty in probability specification and less (resp. more) heterogeneous production functions.

Figure 1: Individual firms' input coefficients *pdf*'s vs. the aggregate level *pdf*.



Note: Individual firms' input coefficients *pdf*'s are represented by blue, dashed lines, and the relevant aggregate *pdf* is illustrated by the bold orange, solid line.

The question now is how to derive the distributions of input coefficients in practice? Ideally, of course, the firm-level data that are, for example, used by statistical agencies in order to derive point estimates of IO coefficients can readily be employed to observe the distributions empirically. However, such data are rarely available to the wide audience of (IO) researchers, while conducting the relevant large-scale firm-level surveys is extremely costly. This might partly explain why the full *pdf* approach has not been empirically applied so far. Thus, Jackson (1986) adopts another feasible (more practical) option instead: using 1972 Use table of the US that distinguishes between 355 sectors, the author first calculates production coefficients, which then he treats '*as though* they were actually firm coefficients' (p.519) in compiling 28 aggregate sectors. The author reports the number of observations for aggregate industries ranging from 9 to 40. Both raw coefficient frequency distributions and distributions for coefficient weighted by output levels (i.e. aggregate distributions) are constructed, which show that the overwhelming majority of the obtained frequency distributions resemble non-symmetric, skewed to the right distributions, similar to the aggregate *pdf* illustrated in Figure 1 above. The author provides a simple, but quite instructive, example of how an increase in demand in one sector results in different values of input coefficients of all aggregate industries over *different rounds of spending*, where the exact values of IO coefficients are determined by the combinations of firms affected by the initial and subsequent changes in input demands. Therefore, it becomes evident that 'the probability of observing any one coefficient appropriate to a given round of spending will be a function of the distribution of firm coefficients, the relative importance of each firm to the industry …, and the nature (distribution) of direct and indirect demand during that particular round. Hence, for all rounds of spending, the full *pdf* coefficient representation is the best available' (p.520-522).

Compared to the point-valued estimates, input coefficient *pdf*'s are indeed more flexible in their representations of the microlevel components. This is also the case because 'some amount of input factor substitutability within firms is already captured by the industry's column coefficient *pfd*'s' (p. 523). In his simulation exercises, Jackson (1986) generates an aggregated 20 sector model from the 355 industry-by-industry table. Forty percent of the obtained IO coefficients distributions with low sample variance become point estimates, while parameters of the theoretical gamma distribution are computed and used for the rest of the coefficients. Forty-eight input coefficient matrices are generated from these distributions (2 draws are ignored because some column sums in these draws exceeded unity), then 48 Leontief matrices, output multipliers and output values are computed. Some of the main results of the simulations are the following: (a) the distributions' skewness statistics are all positive; (b) outputs are 1.56 times more variant than their respective industry multipliers (based on comparison of coefficients of variation), suggesting that primary dependence on multipliers in impact analysis is likely to underestimate the expected variation around gross output vectors; and (c) there is no consistent relation (over- or under-estimation) between the expected values of outcomes from the full *pdf* simulation and their point-estimate counterparts.

All in all, we emphasize that indeed the full *pdf* approach is a quite general explicit representation of uncertainties in an IO framework that is empirically grounded, where all possible values of input coefficients with the associated probabilities are taken into account. Wibe (1982) conducted one of the first studies that used large firm-level data of all Swedish industrial plants in 1979 and concluded that 'a study of the exact distribution of the firms affected by a marginal change in final demand is of great importance for the users of I-O statistics' (p.70). In particular, Wibe showed that more profitable firms (or establishments) within a given industry were more labor and material efficient due to their production practices than the less profitable firms, implying that if only the lowest profitable firms are affected, 'the employment consequences will be 47% greater than the average outcome' (Wibe, 1982, p.70). In essence, this finding confirms the importance of the full distribution approach. It should be also noted that the full-distribution approach does not require additional data given that the same firm-level data traditionally sampled, e.g. by statistical agencies, in order to build IO tables, could be readily used. The traditional way of using such data in deriving point estimates of average coefficients definitely leads to a loss of useful information, in particular, on distributional characteristics of the coefficients or combinations of coefficients that are the basis of the full *pdf* approach. Therefore, it still remains to be seen whether or not this line of research will be taken up in practical applications.

6. MONTE CARLO ANALYSIS

Monte Carlo analysis is another widely used tool in IO research that continues to be of crucial help in carrying out diverse IO-related sensitivity or uncertainty analyses. Several studies use more than one technique in discussing the relevant uncertainty issues. Thus, papers that have already been surveyed in the previous sections in relation to one or all of the tools used (e.g. ten Raa and Steel, 1994; Kop Jansen, 1994; Jackson, 1986) will not be further discussed here.

Quandt (1959) investigated the probabilistic properties of the solution (gross outputs) of a Leontief system by means of a large number of sampling experiments, claiming that '[i]n certain special cases it may be possible analytically to derive the distribution of the solution, but in general this does not appear to be possible' (p.296). For this purpose, a three-sector example is considered. Errors in input coefficients are assumed to be independent, and 11 different three-valued discrete distributions for errors are examined. Comparing the moments of 100 samples of random input matrices and the relevant solutions using Spearman's rank correlation test, the author finds that 'the skewness of the original distribution tends to be transmitted to the distribution of the solution' (p.300). Quandt then fits lognormal distributions to each of the eleven frequency distributions by methods of moments, compares the fitted distributions with the observed ones using a Chi-squared criterion, and concludes that 'the lognormal distribution provides a fairly adequate description of the distribution of the solution, irrespective of the distribution of the original errors' (p.304). The practical implication of this finding, according to Quandt, is that one can use the critical values of the lognormal distribution to establish confidence intervals for the estimated gross outputs.

A number of simulation studies examined the impact of errors in the structural parameters of *regional* IO analyses with the aim of guiding practitioners on optimal allocation of scarce resources for building regional IO tables. For example, Stevens and Trainer (1980), and Park, Mohtadi and Kurbursi (PMK, 1981) carry out simulations by introducing multiplicative errors to the components (except final demand) of the true regional IO model and compare the simulated IO multipliers with the true ones.[20] Errors are normally distributed draws with mean 1.0 and standard deviation of one-half of the percentage errors of 10, 20, 30 or 40 percent of the original coefficients and are multiplied by the original IO coefficients in order to obtain new coefficients. Whereas in the first study true IO matrices were hypothetically generated, PMK (1981) used the Utah 1963 IO table as the true table. One of the main conclusions of these studies was that errors in *regional purchase coefficients* (RPCs - elements of the diagonal matrix $\hat{\mathbf{r}}$ in a regional IO model of $\mathbf{x} = (\mathbf{I} - \hat{\mathbf{r}}\mathbf{A})^{-1}\hat{\mathbf{r}}\mathbf{f}$) contribute far more to inaccuracies in calculated outputs and multipliers than errors in the technical coefficients matrix $\mathbf{A}$.

[20] Park, Mohtadi and Kubursi (1981) provide also analytical results, following Park (1973). However, their complicated form did not allow making statements on the relative importance of the different types of errors, hence the authors perform simulations.

Garhart (1985), however, raised concerns about the practical implication of these results, essentially meaning that one should not be too concerned about technical coefficients, while more efforts and funding needed to be put into the accurate estimation of RPCs in doing regional IO analysis. Garhart noted that one of the explanations of this result is due to the use of a multiplicative error structure, since then errors will be larger if applied to larger coefficients (such as RPCs compared to the technical coefficients) and thus having more impact on the outcomes. On the other hand, 'application of purely additive error structure would bias the analyst to reach the opposite conclusion' (p.356) as the errors would be independent of the size of the coefficients to which they are applied. Therefore, in his simulations, Garhart (1985) used a hybrid error structure, combing multiplicative errors with the additives error components, and found that results are sensitive to such choice of error structure. He concludes that 'labor inputs should be estimated carefully, and no important component of the model should be neglected, including the interindustry technical coefficients, as has been previously suggested' (p.365). This conclusion is consistent with Jensen and West's (1980) experiments on 14 empirical IO tables for regions ranging from small rural region economies to the whole of Australia; by eliminating 5% of the coefficients at a time and recalculating output, income and employment multipliers they found that setting to zero larger coefficients has far much larger impact on multiplier accuracy than replacing smaller coefficients with zeros. This implies that for the overall accuracy of multipliers and IO analysis, it is important that the large coefficients in IO tables are estimated as accurately as possible, while one should not be too much concerned about the accuracy of smaller entries in the tables.

In their Monte Carlo simulations, Bullard and Sebald (1988) show that their analytical maximum error tolerances derived in Bullard and Sebald (1977, see Section 2) do not reflect the observed precision of IO calculations. In their analysis of the 1967 US table, the authors demonstrate that 'input data uncertainties combine or cancel one another in a manner that holds error magnification to acceptable levels, [and the] results seem to be insensitive to the level of aggregation' (Bullard and Sebald, 1988, p.711).

Several papers investigated the issue of the bias in the Leontief inverse and/or IO multipliers from Monte Carlo analysis point of view. In his simulations, Roland-Holst (1989) first draws transaction flows errors independently from a zero-mean Normal distribution, adds them to the respective observed transactions values, derives 'random' Leontief inverse, repeats this procedure *n* times, computes the average value of the obtained 'random' elements from all *n* simulation runs (i.e. across all individual multiplier matrices), and finally statistically compares the difference between the last and the appropriate observed transactions flows. From his rather extensive experiments, involving sample sizes *n* = 10, 20, 30, 50, 100, Social Accounting Matrices (SAMs) of Botswana and Korea, and different combinations of exogenous and endogenous number of institutions of the US SAM, Roland-Holst (1989) concludes that disaggregated multiplier estimates are unbiased, implying that 'transactions tables, if they are observed with well-behaved (in this case normal) measurement error, will yield multiplier estimates centered on their true distribution' (p.720). Additionally, the author finds that if the Monte Carlo sample is large enough, the disaggregated multiplier estimates can be significantly more stable than their transactions matrix counterparts.

In a similar study, Dietzenbacher (2006) suggested that Roland-Holst's (1989) sample was too small. Running analogous Monte Carlo simulations, assuming the transactions are unbiased and independently, normally distributed and using 30/34-sector IO tables of ten OECD countries and 128-sector IO table for the Netherlands, Dietzenbacher (2006) concludes that in practice the 'multiplier estimates *are* positively biased but the biases are negligibly small. The empirical relevance of this conclusion is that in carrying out the typical interindustry calculations we may proceed as if the multipliers are unbiased' (p.775).

Rueda-Cantuche, Dietzenbacher, Fernández and Amores (RDFA, 2013) take the prior analysis one step further by imposing stochastics on supply and use tables (SUTs) rather than on IO tables. The underlying reason is that intermediate transactions, though historically obtained from published IO tables, nowadays are constructed from SUTs using certain technology or sales structure assumptions. Without going into all the details,[21] RDFA (2013) carry out Monte Carlo simulations using 2006 SUTs of Spain, Italy, the Netherlands, Germany and Finland, and confirm the findings of Roland-Holst (1989) and Dietzenbacher (2006) for randomized intermediate flows that multipliers' biases do exist but are 'negligibly small'.

Uncertainties in all components of the IO model within a multi-region IO (MRIO) setting are examined in a case study on the UK's carbon footprint by Lenzen, Wood and Wiedmann (2010). They first perturb the elements of the UK-MRIO table, gross outputs, and sectoral CO_2 intensities according to their respective *relative standard*

[21] The procedure of SUTs randomization is more complicated than IO flows randomization. For example, the new issues that need to be dealt with include the following: (a) the randomized SUTs need to be made consistent with each other, and (b) one has to make a choice of a technology or sales structure assumption in order to construct IO table from SUTs.

deviations (RSDs, i.e. variable change divided by its base-year level), then calculate the perturbed CO_2 multipliers, repeat the procedure 5000 times per year, and compare the perturbed and unperturbed multipliers in order to generate distributions of CO_2 multipliers' RSDs. The means of these distributions are then considered as actual multipliers' RSDs. Combining these with RSDs of final demand using error propagation, the authors arrive at uncertainties (SDs) for the carbon footprint components. In particular, their 'uncertainty analysis shows with statistical significance that … against popular belief, the carbon footprint of the UK has been increasing rather than decreasing' (p.60).

There are other studies that make use of Monte Carlo simulations to better understand IO data uncertainty implications. For example, Lenzen (2001) examines the effect of uncertainties in production factor coefficients and IO coefficients on labor and energy multipliers. Wilting (2012) carries out an extensive uncertainty analysis focusing on the Dutch carbon footprint. He shows that uncertainties in the model coefficients resulted in a relatively low degree of uncertainty in the total Dutch carbon footprint, while uncertainties in the carbon emissions allocated to regions, sectors and products were larger. Additionally, the author concludes that in certain cases applying a partial MRIO analysis is justifiable, in which case in the inter-regional input matrix 'most of the off-diagonal blocks are left out [and] only the import blocks of the region under consideration have non-zero elements' (p.143).

At the global level, Karstensen, Peters and Andrew (2014) estimate uncertainties in economic data, emission statistics and metric parameters in order to understand how they propagate from regional and sectoral production- and consumption-based emissions to the global temperature change estimates. They find that '[u]ncertainties in the final results are largely dominated by the climate sensitivity and the parameters associated with the warming effects of CO_2' (p.1014). They find that at the global and national levels, economic data have a relatively small impact on uncertainty, and thus their 'results suggest that consumption-based national emissions are not significantly more uncertain than the corresponding production-based emissions' (p.1014). Similarly, a Monte Carlo uncertainty analysis of significance of time variations of the total forward and backward linkages (interpreted, respectively, as average distance of industries along the global output supply and input demand chains with respect to households, government and investors as users of final products and as providers of primary inputs) within the global IO framework is carried out in Miller and Temurshoev (2015).

When it comes to the discussion of the integrated econometric and IO models, dealing with the issue of uncertainty becomes more challenging due to higher dimensionality of these types of models. In this respect, Rey, West and Janikas (2004) focus on three sources of uncertainty: IO coefficient uncertainty, econometric model parameter uncertainty, and econometric disturbance term uncertainty.[22] Through a series of Monte Carlo simulations the authors analyzed the relative importance of each of these components and tried to understand how their interactions propagate through the integrated model on the distributions of the endogenous variables. They could not come to a decisive conclusion on which source of uncertainty is the most important one, and conclude that '[i]nstead, that answer is conditioned upon the focus of the analysis and whether the industry specific or macro variables are of central concern' (p.275).

Usually in IO analysis, the well-known price and quantity IO systems are analyzed in complete isolation from one another, which rightfully raises a lot of criticism, especially from outside observers, because obviously in reality prices and quantities interact. Following ten Raa and Shestalova (2015a), ten Raa and Shestalova (2015b) account for interdependencies of the two dual systems in equilibrium analysis by introducing *complementarity* into IO analysis. In short, the complementarity conditions imply that (a) the price of a product in excess supply will be zero, and (b) the supply of a commodity incurring loss to its producers will be zero. In this setting, prices and quantities of commodities and factors of production are endogenous, but the structure of the economy, represented by e.g. intermediate inputs and factor coefficients, consumption proportions and factor supply, are exogenous. Thus, the setting readily allows the use of Monte Carlo analysis of the sensitivity of the outcomes of interests to changes in the components of the structure of the economy. Ten Raa and Shestalova (2015a) study the impact of caps and international trade in CO_2 emission permits on consumption expansion factors for Belgium, Denmark and Spain, while in ten Raa and Shestalova (2015b) the relevant uncertainty analysis is carried out. In particular, assuming that the distributions of the emission caps per country are one-sided and uniform, the authors run the model 100 times and find that the equilibrium expansion

[22] West and Jackson (2014) present a Simulating Impacts on Regional Economies (SIRE) model, which, according to the authors, occupies an intermediate position between the IO and CGE models. In this setting also the importance of uncertainties is mentioned, in particular, the authors note that 'presentation of impacts assessments from price-sensitive models is *incomplete* without a presentation and assessment of the role of elasticities in generating results' (p.149, emphasis added). This is crucial observation because in the presence of large import substitution elasticities, the SIRE-based multiplier estimates could be larger than those based on simple IO models, contrary to the widespread belief.

factors have relatively small variance (or narrow confidence intervals) for all three countries.

7. BAYESIAN APPROACH

The Bayesian approach in econometrics and, in general, in data analysis has become a popular tool in recent decades, partly explained by the fact that due to the rapid development of computer technology, running complex Bayesian posterior simulators that caused computational problems in practice earlier, becomes rather easy. Bayesian estimation is based on sound probability theory and has many advantages. For example, (a) results are presented in terms of intuitively meaningful posterior densities, (b) any non-sample information can be effectively used via priors' density specifications, and (c) marginal posterior densities reflect all the parameter uncertainty in a model and are not conditioned on point estimates of parameters of no primary interest (i.e., nuisance parameters). In fact, dealing with nuisance parameters in a general setting is one of the major problems frequentist researchers face. Thus, applications of Bayesian tools can have a major beneficial influence in the field of stochastic IO analysis as well. At this stage, one can say that Bayesian IO-related research is still in its infancy, but we expect it to be a very fruitful tool for IO analysis in the future.

Bayesian approach has been recently adopted in the compilation of national accounts, based on the work by Magnus, van Tongeren and de Vos (2000), Magnus and van Tongeren (2002), and Danilov and Magnus (2008). The Bayesian approach facilitates estimation of the variables, indicator ratios between the variables, and reliability intervals of the estimates. The approach takes into account all available information together with its (available or assumed) prior precision; further, it is able to deal with multiple priors for the same variable, and estimates all variables and indicator ratios with their reliability values. The last consideration is an important addition to the literature, as variables estimates in national accounts are usually point estimates. This approach led to what is now called the *Bayesian System of National Accounts* (BSNA) framework that has already been applied to six Central-American countries (Costa Rica, El Salvador, Guatemala, Honduras, Nicaragua, and the Dominican Republic), 'as part of a project sponsored by the Netherlands Organization for International Cooperation in Higher Education (NUFFIC) in cooperation with the Instituut voor Ontwikklingsvraagstukken at Tilburg University, The Netherlands, and the Consejo

Monetario Centroamericano' (see van Tongeren and Magnus, 2012, p.278). This paper also presents for the first time an empirical application to a large dataset of Guatemala (2500 variables). Thus, the Bayesian method, similar to the conventional national accounting approaches, deals with 'reconciliation' or 'integration' of data, but differs from the last in several respects. 'First, all conditions of conventional estimates are formalized: identities are explicitly included and ratios are introduced as priors; second, reliabilities of data and ratio values are reflected in well-defined prior variation coefficients; third, the system is simultaneous rather than sequential; and finally, updating the system when new information becomes available is easy and fast, and does not require changes in the compilation method' (van Tongeren and Magnus, 2012, p.278). In addition, '[t]he method also facilitates the simultaneous use of indicator ratios in compilation and *analysis*' (Magnus, van Tongeren and de Vos, 2000, p.347, emphasis added). For IO analysis, the implication is that the estimated supply and use tables from the Bayesian approach will have reliability intervals element-wise, and thus can be readily used in subsequent analysis, e.g. sensitivity analysis. Or alternatively, the method itself can be used, alone or in combination with existing ones, in order to construct IO tables, again with uncertainty values attached to each element. Such an attempt is already made by Lugovoy, Polbin and Potashnikov (2014) who update IO tables for selected EU countries and SUTs for Russia using Bayesian methods. Their experimental results and comparisons with other existing updating methods confirm the usefulness of such approach.

The problem of balancing IO data from the Bayesian perspective is considered by Rodrigues (2014). The author states that considering all available information in dealing with the data balancing problem implies accounting explicitly for the first and second moment constraints that include prior information about the values of the data (best guess) and their uncertainties. Rodrigues (2014) then derives an analytical solution of the data balancing problem by application of cross-entropy minimization subject to the first and second moment constraints, and shows that the conventional data balancing methods, such as generalized least squares, weighted least squares and biproportional methods, are particular cases of the proposed method.

Rickman (2001) presents a Bayesian integrated IO and econometric model of the state of Oklahoma. Comparing seven models in terms of their performance in forecasting industry employment, the author, for example, concludes that '[o]n average, a Bayesian model produced the lowest MAPE [mean absolute percentage error] across the thirty employment sectors and five forecast horizons' (p.241).

On the basis of ten Raa and Rueda-Cantuche's (2007) equation (6), Temurshoev (2012) estimates IO multipliers, but with a difference that instead of traditional econometric techniques, a Bayesian approach is adopted. The reasoning behind this choice is twofold. First, when economy-wide SUTs are used, reducing product dimension to generate sufficient degrees of freedom may lead to a potentially severe loss of information, in particular, such aggregation will result in ignoring information on the heterogeneities of the aggregated product-industry relationships. Secondly, it may very well be the case that the estimates of IO multipliers when a simple OLS is used, are lower than their economically plausible lower bounds. Using international SUTs from the World Input-Output Database project, Temurshoev (2012) quantifies and presents the development of the product-level global carbon dioxide emission multipliers for 40 countries and 59 products over the period of 1999-2009.

Finally, Temurshoev (2015a) proposes a new framework for estimating product-level global and interregional feedback and spillover factor effects directly from interregional/international SUTs. The framework allows for SUTs to be rectangular and gives a possibility of taking into account the inherent data uncertainty problems. Using a Bayesian approach, the author estimates and presents the global and intercountry feedback-spillover output effects at the world, country and product levels for the period of 1995-2009. It is found, for example, that the overall intercountry feedback-spillover effects contribution to the global output effects was, on average, 7.9% in 1995, 11.5% in 2008 and 9.8% in 2009 (this decrease is due to the 2008-2009 crisis). Such positive (average) changes of the feedback-spillover-to-global output effects ratios were found for 58 (out of 59) products.

8. OTHER TECHNIQUES

In this section, we very briefly discuss or simply mention other tools, some of which are less known within the general IO community, that also have been used in uncertainty treatment in IO analysis. One such method has been developed by Nijkamp, Oosterhaven, Ouwersloot and Rietveld (1992) who combined ordinal (qualitative) data techniques and error measurement in IO analysis and refer to this combined method as ‘ordinal input-output analysis’ (p.416). In order to estimate IO coefficients, this method starts by using ordinal information in the sense of input coefficients ranking, where such cases as small or large degree of differences between consecutive coefficients, ties or incomplete rankings can be explicitly taken into account. Subsequently, this ordinal information is transformed into cardinal data by using a minimum number of assumptions on the underlying probability distributions of the IO coefficients (e.g. in the absence of additional information, the well-known principle of insufficient reason would imply using uniform distribution), while a random number generator makes the core of the stochastic method that is used to generate an empirical distribution of interest. The authors conclude that '[t]he main advantage of *ordinal input-output analysis* is the integration of input-output table construction and the determination of standard errors into one framework' (p.416, emphasis added) and the last 'can be used as a tool for error measurement' (p.408).

The notion of a balanced growth path is of particular interest in the analysis of dynamic IO models. Ćmiel and Gurgul (1996, 1997) introduce backward, forward and mixed time lags into the IO dynamic model, and study some of their statistical properties. Specifically, they focus on the random balanced growth factor within each setting, which 'enables [one] to take into account the uncertainty that is always present in empirical investigations of real economic processes' (Ćmiel and Gurgul, 1996, p.138). The authors derive distribution functions and approximate formulas for the stochastic characteristics of the random balance growth factors for the so-called forward-lag-type, backward-lag-type and mixed-lag-type IO models.

Other approaches attempt to account for inherent IO data uncertainties in their final outcomes include such tools as *interval arithmetic* (see e.g. Jerrell, 1997; Barker and Rocco S., 2011), *fuzzy systems theory* (see e.g. Beynon, Munday and Roberts, 2005; Beynon and Munday, 2007, 2008; Díaz and Morillas, 2011), and *grey systems theory* (see e.g. Wu and Chang, 2003; Li and Liu, 2008). Similarly, to measure global robustness of the outcomes of empirical IO models with respect to errors in IO coefficients, Wolff (2005) applies the *theory of robust systems* (or theory of norms) in linear algebra. Presenting the main features of these techniques is apparently beyond the scope of this chapter, and thus the interested reader is referred to the mentioned references.

9. DISCUSSIONS AND CONCLUDING REMARKS

In this study, we have surveyed the major contributions to the field of stochastic input-output (IO) analysis broadly defined. The literature was discussed rather extensively according to the specific tools used in dealing with the inherent IO data uncertainty problems. Within each such blocks a sort of history of thought approach has been adopted as well in order to trace the development of research on a particular considered topic, which also reveals the state of the art findings. Wherever possible, the connections between the results of the individual studies or tools used were spelled out and clarified.

One of the main findings of this survey is the discovery of Fox and Quirk (1985) work on derivation of the general formula of the probability density function of the Leontief inverse for a given input coefficients density. This was a surprising finding because until now this contribution was entirely unknown to IO economists and other researchers dealing with the uncertainty issues of the IO systems.

Referring to stochastic IO analysis and, in particular, to a better understanding of IO coefficients/transactions variations and distributions within the individual industry structures, West (1990) noted the following. 'It would be indeed unfortunate if the trend toward less survey data makes this type of analysis, which is just in its infancy, unfeasible in the future, since this avenue of research is potentially extremely useful and important in understanding of the nature and operation of regional economies, in addition to making the model much more attractive to the potential user' (p.116). Now looking backwards, it seems that West's fear has indeed largely became reality, at least given the state of the art regarding our understanding of distributions or variations of the individual IO coefficients and transactions, in particular, of those which are important in terms of their system-wide impact (this view is essentially consistent with the full distribution approach to uncertainty). Moreover, given the explicit formulation of the Leontief inverse probability density function due to Fox and Quirk (1985), it is now even more important to have empirical basis for making assumptions regarding input coefficients density. Hence, a very fruitful research agenda in this respect would be carrying out an extensive empirical assessments of the distributional characteristics of establishment level IO coefficients and transactions, already forcefully emphasized by Jackson (1986, 1989). This would require intensive collaborations between statistical offices and IO researchers that would finally benefit all parties since then the richness of information of available firm level data behind published IO tables would be used efficiently, which otherwise is inevitably lost if used for derivations of point estimates of the average coefficients and transactions only.

In terms of Jensen's (1980) *partitive-holistic accuracy* framework in the context of IO tables, 'the accuracy of the table would be judged, not on the accuracy of its separate parts [i.e. partitive accuracy], but on its ability to represent the size and structure of the economy in general terms [i.e. holistic accuracy]' (p.143). Thus, Jensen advocated that the holistic accuracy is (or should be) the only tenable concept in practice. Applying this framework to the uncertainty treatment in IO tables and analysis would mean that one should not worry too much about the accuracy of each and every cell in the obtained IO coefficients or transactions matrix but rather an adequate care should be directed towards the accuracy of, for example, only 'significant in some sense' transactions, sectoral IO multipliers or forecasts at the higher than IO cell-specific levels. Looking into our survey, it seems that the holistic concept of accuracy has been, at least partially, recognized in some of the more recent stochastic IO analysis studies with the focus on uncertainties in IO multipliers, while the earlier contributions mainly addressed the impact of errors or uncertainties in the individual input coefficients. This could be also due to the finding of error-compensating properties of the IO system in practice, discovered first in deterministic error analysis literature, notably by Evans (1954) and Christ (1955), and consequently repeatedly confirmed in other studies using different tools. However, a word of caution regarding a sole focus on IO multipliers is in order here: we can generate similar IO multipliers from a variety of IO tables, yet when we conduct impact analysis, we will obtain different results (see e.g. Hewings, 1977; Harrigan, 1982). Thus, having *actual* IO tables and/or SUTs is of crucial importance, even if their complication is costly both financially and time-wise. Therefore, also in the spirit of Jensen's ideas, the focus needs to be on the integrity of the IO table and/or SUTs to represent the economy.

It follows from this overview that practitioners have widely used Monte Carlo analysis in their sensitivity or uncertainty analyses, in particular in recent decades. The findings of the similarly large literature on random error analysis or probabilistic approach were more of theoretical nature, and have largely remained at this level so far. For example, the observation that the first-order approximating formulas of the bias and/or sensitivity of the Leontief inverse elements have never been used, to our best knowledge, in practice is most probably due to the fact that their implementation requires prior knowledge of the covariances between all the underling IO input coefficients in case of ten Raa and Kop Jansen's (1998) formulae, or transactions in case of the McCamley, Schreiner and Muncrief (1973) variance formula for IO multipliers. Apparently, this kind of information is largely missing and requires carrying out expensive firm-level surveys, or better as mentioned above, effective collaborations among statistical agencies engaged in IO and SUTs compilation and IO researchers are necessary.

At the same time, we also observe that new approaches are being applied to problems in stochastic IO analysis. These new lines of research that also seem to us promising in the future include the Bayesian approach to data analysis and equilibrium analysis in IO framework based on complementarity concepts (ten Raa and Shestalova, 2015a, 2015b). We expect Bayesian techniques to be useful in this field as they have already proved to be extremely beneficial and successfully applied in other fields of sciences. Equilibrium analysis with complementarity tools is similarly an important addition to IO modeling approach in general, because it combines the IO price and quantity systems that many IO economists applied largely in isolation to date, and as such this setting is more suitable for economic or environmental policy impact assessment purposes.

We conclude this chapter by suggesting that IO researchers need to pay particular attention to uncertainty issues. Unfortunately, there are still abundant studies that completely ignore the issue and derive their conclusions solely on the base of the point-estimates.[23] The wealth of knowledge cumulated so far in this respect should be accessed to conduct some sensitivity/uncertainty analysis. Of course, the choice of a technique to be used will depend on many factors, including data availability, the focus of the analysis and so forth. But there is always an option of doing, at least, some rather simple and straightforward sensitivity exercises. For example, similar to Bullard and Sebald (1977), one can perturb certain entries of the input matrix that are relevant for the topic in question, in particular when the focus of the analysis is on emissions or other factors embodied in consumption, and discuss the obtained outcomes' ranges. This should always shed extra light on the results and additionally adds to the credibility of the work itself, given that standard IO models are essentially photographs of real economies and thus issues of uncertainties become critical as one tries to make sense of e.g. the role of economic structure and structural change. However, it should be noted that for economic analysis aiming at assessing various policies using the traditional IO model is not sufficient anymore, and there is the need to base the analysis more on micro-economic foundations so that the resulting model more faithfully reflects the heterogeneity of the economy and various price-quantity interactions in it that the model purports to represent (see e.g. Dixon and Jorgenson, 2013; Kratena, Streicher, Temurshoev and colleagues, 2013; Kim, Kratena and Hewings, 2014).

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[23] It should be noted that this point is also valid with respect to part of the work of the author of this chapter either.

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